Markov Models

Uncertainty and Time

- Often, we want to reason about a ***sequence*** of observations where the state of the underlying system is ***changing***
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Global climate

- Need to introduce time into our models

Markov Models (aka Markov chain/process)

$X \to Y \to Z$

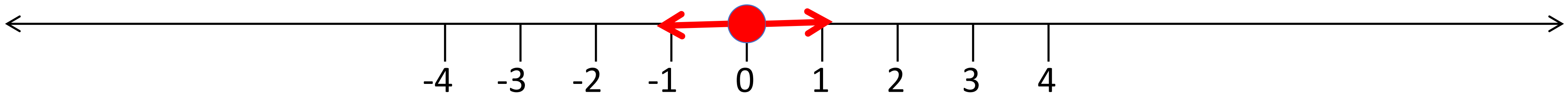
$P(X_t = x \mid X_{t-1} = y) = f(x, y)$

$$X_0 \to X_1 \to X_2 \to X_3 \dashrightarrow$$

$P(X_0)$ $\qquad$ $P(X_t \mid X_{t-1})$

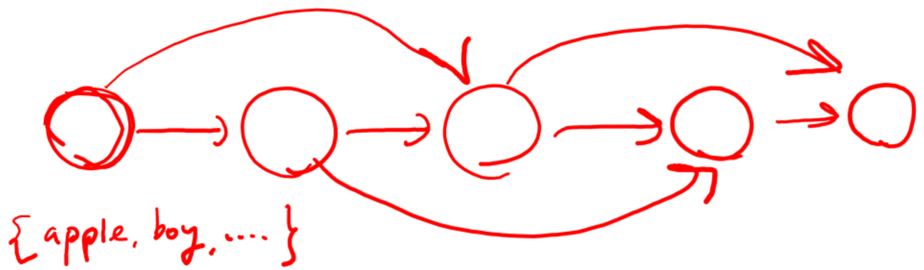
- Value of X at a given time is called the **state**
- The **transition model** $P(X_t \mid X_{t-1})$ specifies how the state evolves over time
- **Stationarity** assumption: transition probabilities are the same at all times
- **Markov** assumption: “future is independent of the past given the present”
  - X_{t+1} is independent of $X_0, \ldots, X_{t-1}$ given X_t

Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial probability: starts at 0
- Transition model: $P(X_t = k | X_{t-1} = k \pm 1) = 0.5$
- Applications: particle motion in crystals, stock prices, etc.

Example: n-gram models



- State: word at position t in text (can also build letter n-grams)
- Transition model (probabilities come from empirical frequencies):
  - Unigram (zero-order): $P(\text{Word}_t = i)$
    - "logical are as are confusion a may right tries agent goal the was . . ."
  - Bigram (first-order): $P(\text{Word}_t = i \mid \text{Word}_{t-1} = j)$
    - "systems are very similar computational approach would be represented . . ."
  - Trigram (second-order): $P(\text{Word}_t = i \mid \text{Word}_{t-1} = j, \text{Word}_{t-2} = k)$
    - "planning and scheduling are integrated the success of naive bayes model is . . ."
- Applications: text classification, spam detection, author identification, language classification, speech recognition

Example: Web browsing

- State: URL visited at step t
- Transition model:
  - With probability p, choose an outgoing link at random
  - With probability $(1-p)$, choose an arbitrary new page
- Question: What is the ***stationary distribution*** over pages?
  - I.e., if the process runs forever, what fraction of time does it spend in any given page?
- Application: Google page rank

Example: Weather

Bayes net $W_1 \to W_2 \to \cdots$

- States {rain, sun}

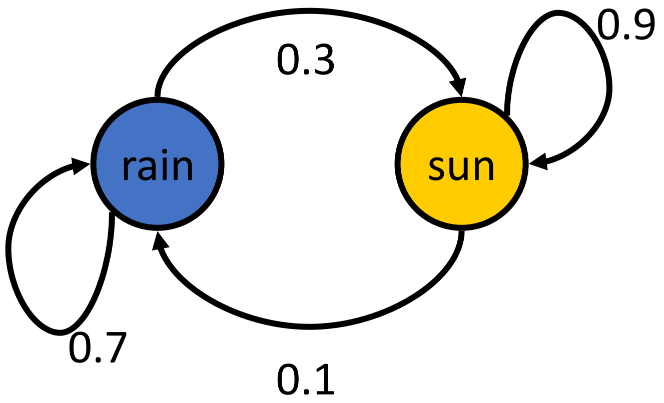
- Initial distribution $P(X_0)$

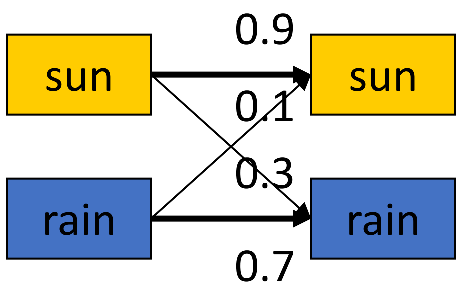
$P(X_0)$	
sun	rain
0.5	0.5

- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Two ways to represent Markov chains





Weather prediction

$$P(X_{t-1}) \xrightarrow{\text{join}} P(X_{t-1}, X_t) \xrightarrow{\text{margin}} P(X_t)$$

- Time 0: $\langle 0.5, 0.5 \rangle$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 1?

$$P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$$
$$= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$$
$$= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle$$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 2?

$$P(X_2) = \sum_{x_1} P(X_2, X_1 = x_1)$$
$$= \sum_{x_1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$$
$$= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$$

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 3?

$$P(X_3) = \sum_{x_2} P(X_3, X_2 = x_2)$$
$$= \sum_{x_2} P(X_2 = x_2)\, P(X_3 \mid X_2 = x_2)$$
$$= 0.66\langle 0.9, 0.1\rangle + 0.34\langle 0.3, 0.7\rangle = \langle 0.696, 0.304\rangle$$

Forward algorithm (simple form)

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \dashrightarrow$$

$P(X_0)$ $\qquad$ $P(X_t \mid X_{t-1})$

What is the state at time t?

$$P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$$
$$= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\, P(X_t \mid X_{t-1}=x_{t-1})$$

Forward algorithm in Matrices

- What is the weather like at time 2?
  - $P(X_2) = 0.6<0.9,0.1> + 0.4<0.3,0.7> = <0.66,0.34>$
- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \underbrace{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}_{P(X_1)} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Stationary Distributions

- The limiting distribution is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^T P_\infty$

  Stationary distribution is $<0.75,0.25>$ ***regardless of starting distribution***

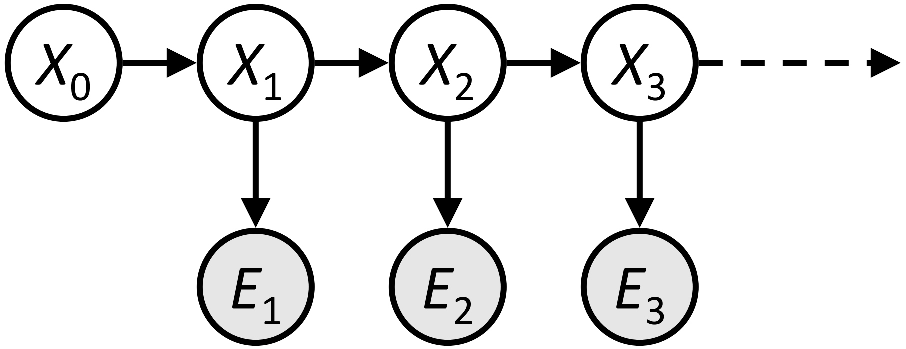
$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \overset{\text{sum}}{p} \\ \underline{1-p} \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

Hidden Markov Models

Hidden Markov Models

- Usually the true state is not observed directly

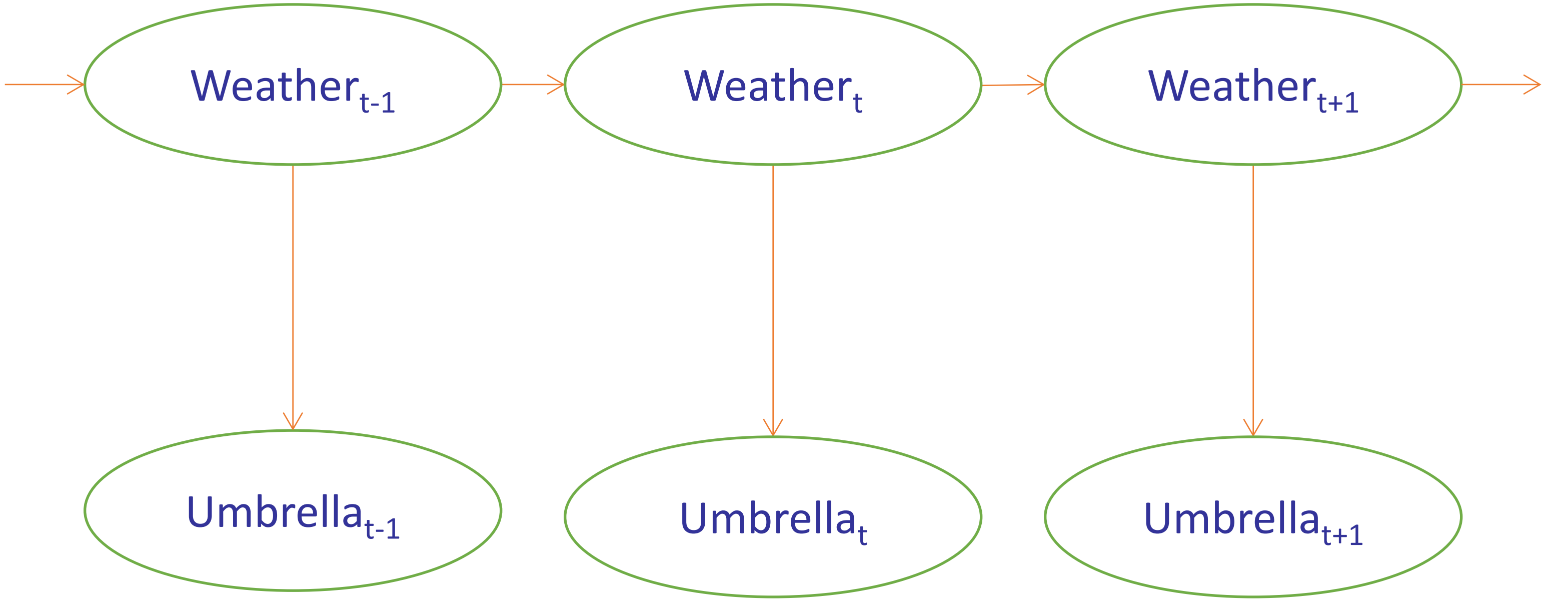
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe evidence E at each time step
  - X_t is a single discrete variable; E_t may be continuous and may consist of several variables



Example: Weather HMM

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

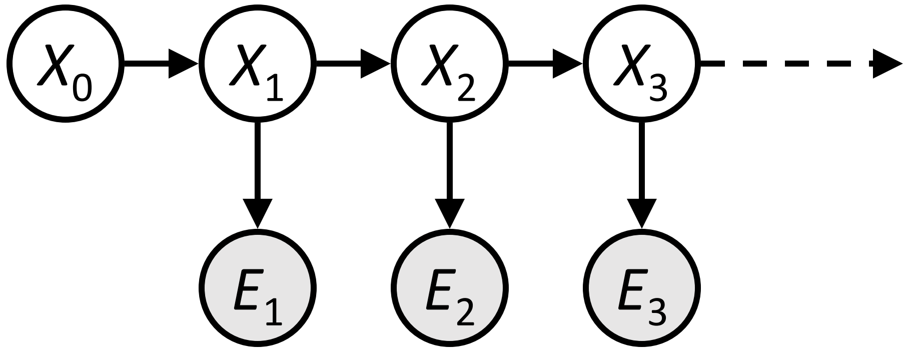
- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t \mid X_{t-1})$
  - Sensor model: $P(E_t \mid X_t)$



W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as probability model

- Joint distribution for Markov model: $P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:
  $P(X_0,E_0,X_1,E_1,\ldots, X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$
- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?



Real HMM Examples

apple

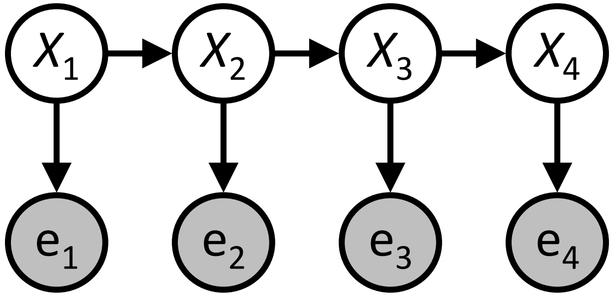
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options

- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Conditional Independence in HMM



Base on D-separation algorithm (see the Bayesian Network slides). Since no two nodes have common children, to test whether $A \perp\!\!\!\perp B \mid C$, we only need to check whether C blocks every path from A to B

For example,

$X_j \perp\!\!\!\perp X_k \mid X_i \qquad \forall\, j < i < k$

$X_j \perp\!\!\!\perp E_k \mid X_i \qquad \forall\, j < i \leq k$

$E_j \perp\!\!\!\perp X_k \mid X_i \qquad \forall\, j \leq i < k$

$E_j \perp\!\!\!\perp E_k \mid X_i \qquad \forall\, j \leq i \leq k, \qquad j \neq k$

Inference tasks

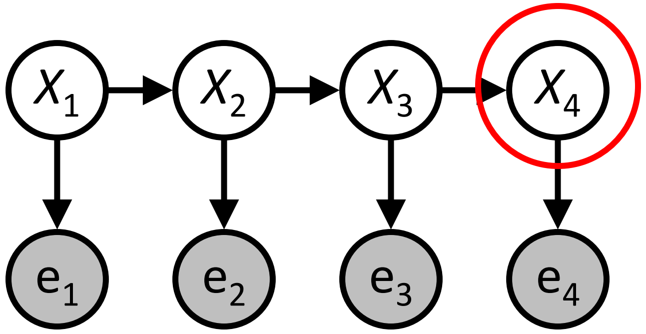
Useful notation:

$X_{a:b} = X_a, X_{a+1}, \ldots, X_b$

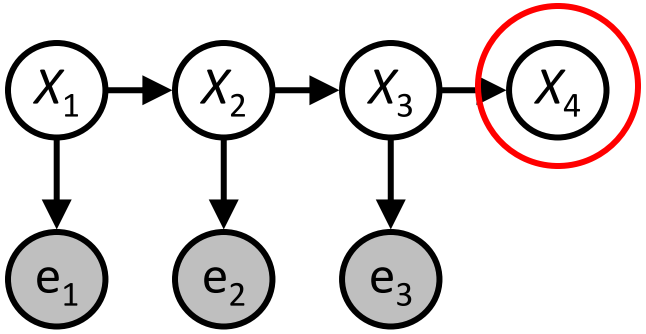
- **Filtering**: $P(X_t | e_{1:t})$
  - **belief state**—input to the decision process of a rational agent
- **Prediction**: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- **Smoothing**: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- **Most likely explanation**: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Inference tasks

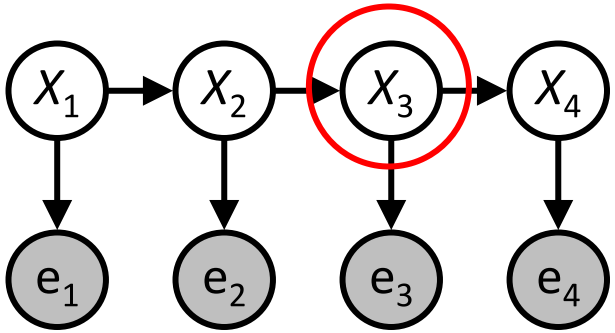
Filtering: $P(X_t|e_{1:t})$



Prediction: $P(X_{t+k}|e_{1:t})$

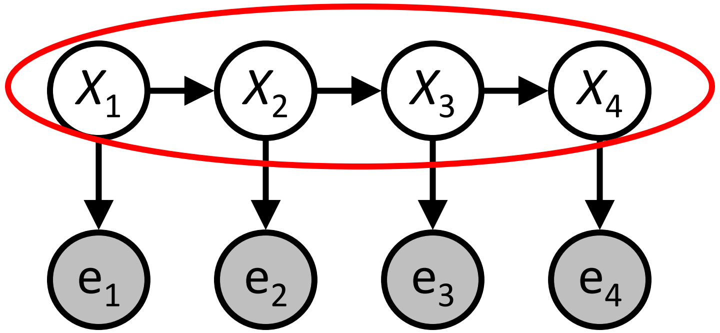


Smoothing: $P(X_k|e_{1:t})$, $k<t$



argmax

Explanation: $P(X_{1:t}|e_{1:t})$



Filtering / Monitoring

- Filtering, or monitoring, or state estimation, is the task of maintaining the distribution $P(X_t|e_{1:t})$ over time

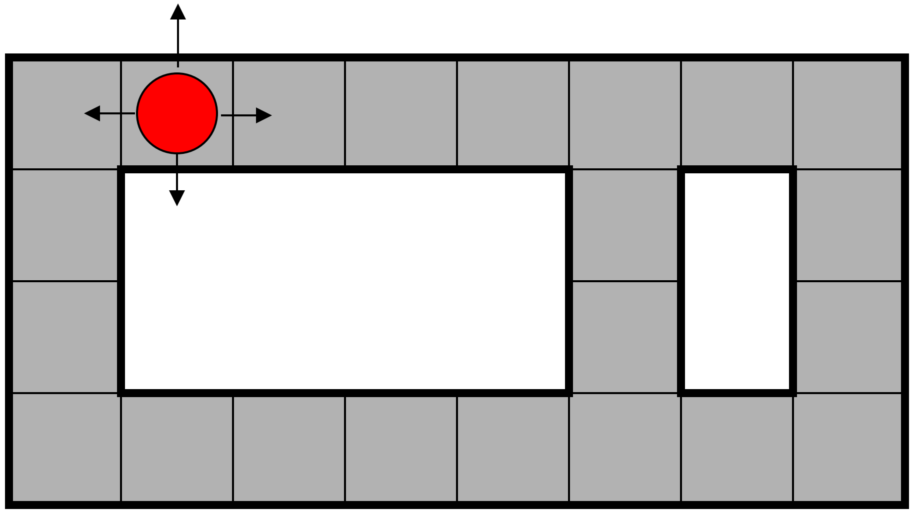
(transition) $P(X_t | X_{t-1})$

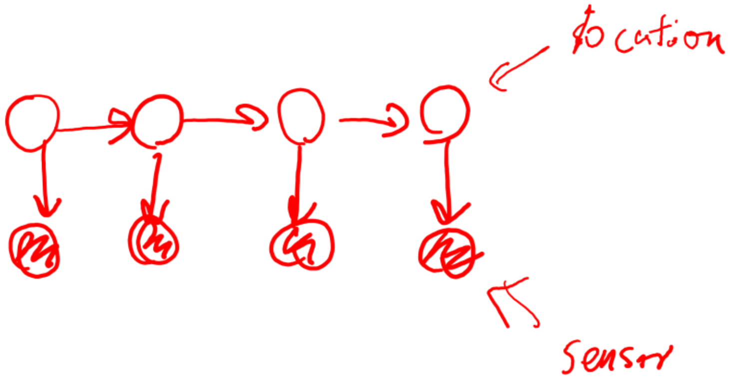
(emission) $P(e_t | X_t)$

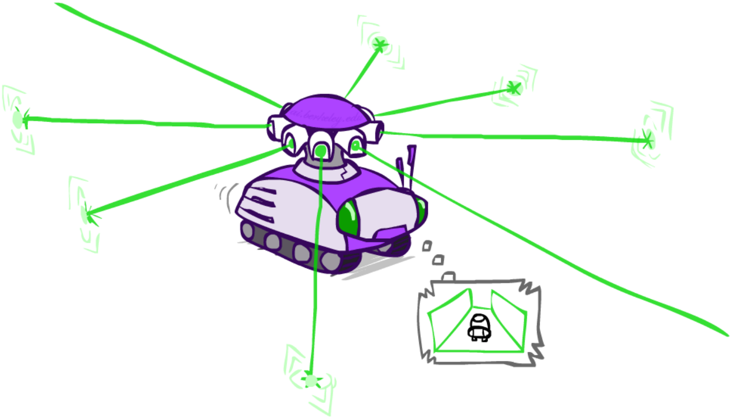
- The Kalman filter (continuous variables, ~~linear~~ dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program.

Example: Robot Localization

Example from
Michael Pfeiffer







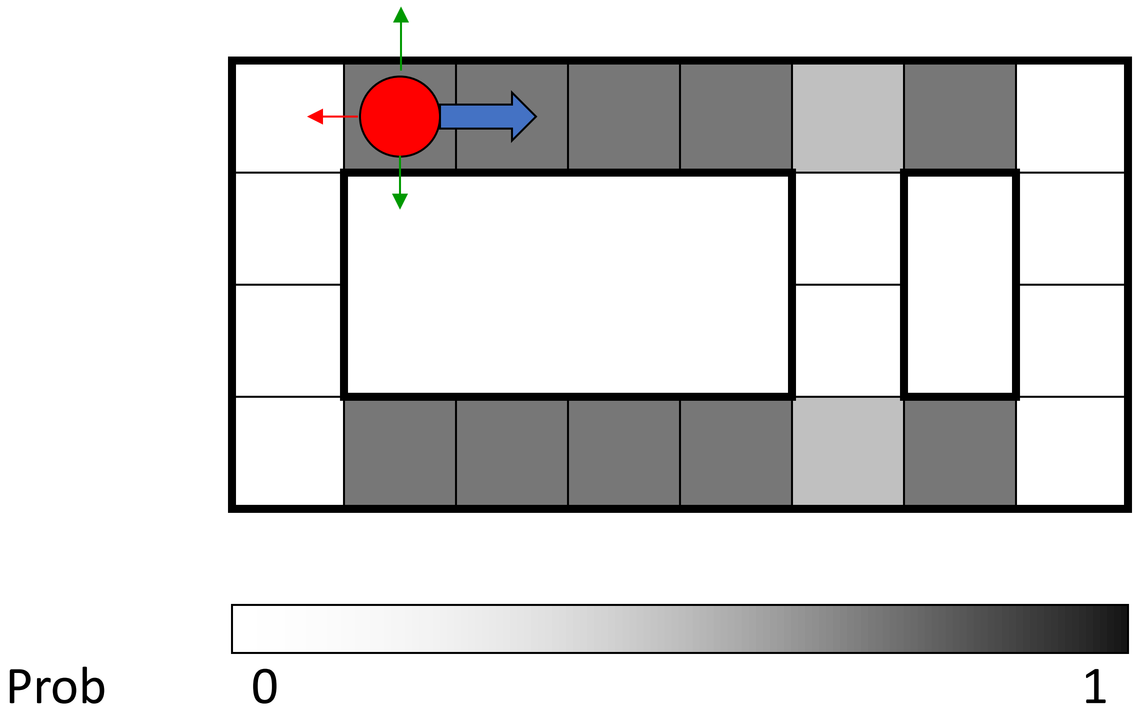
Prob 0 1

t=0

Sensor model: four bits for wall/no-wall in each direction, never more than 1 mistake

Transition model: action may fail with small prob.

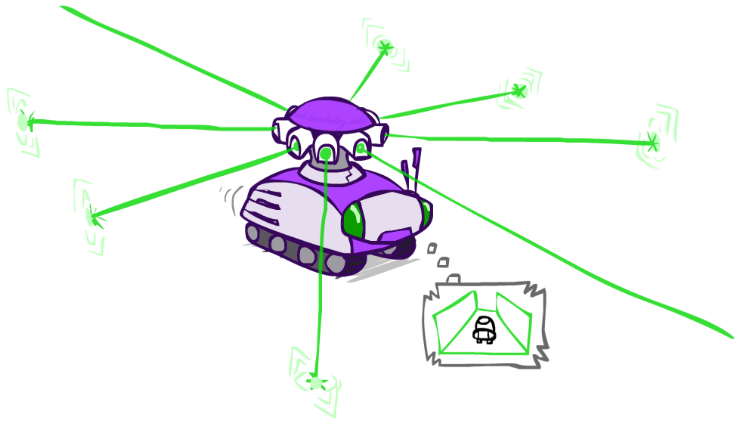
Example: Robot Localization



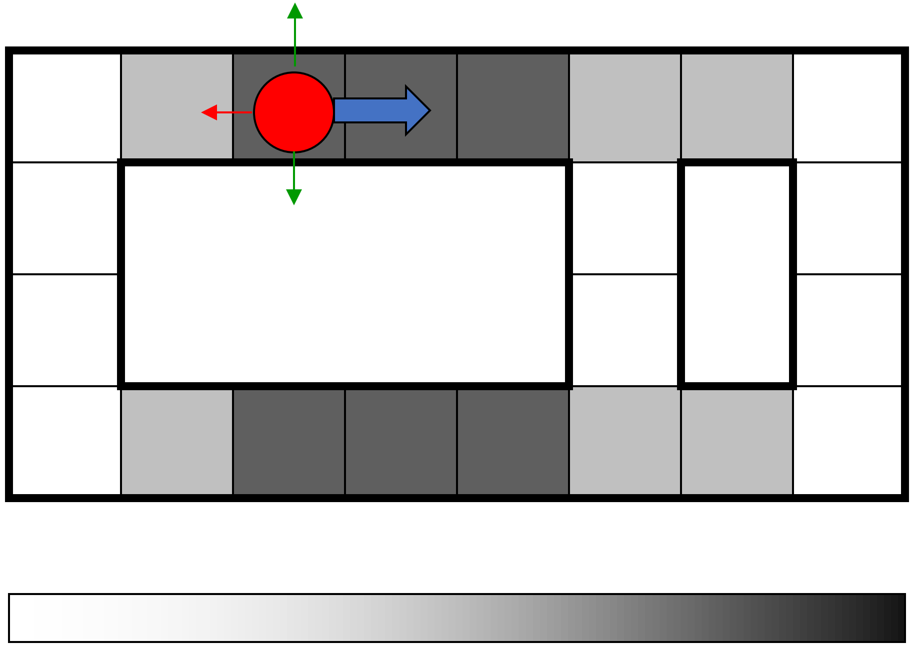
t=1

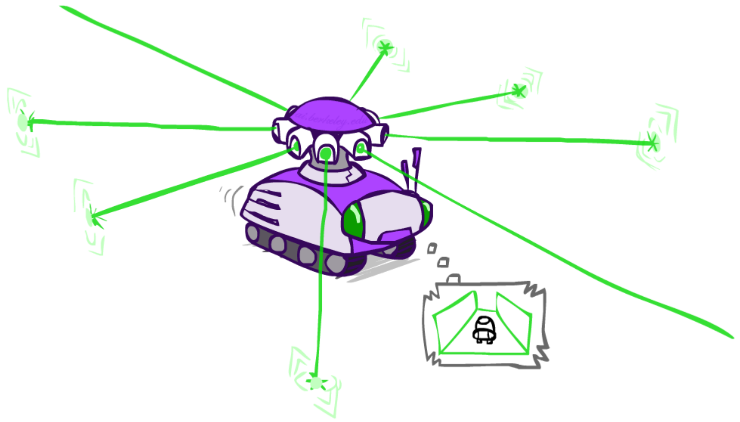
Lighter grey: was ***possible*** to get the reading,

but ***less likely*** (required 1 mistake)



Example: Robot Localization

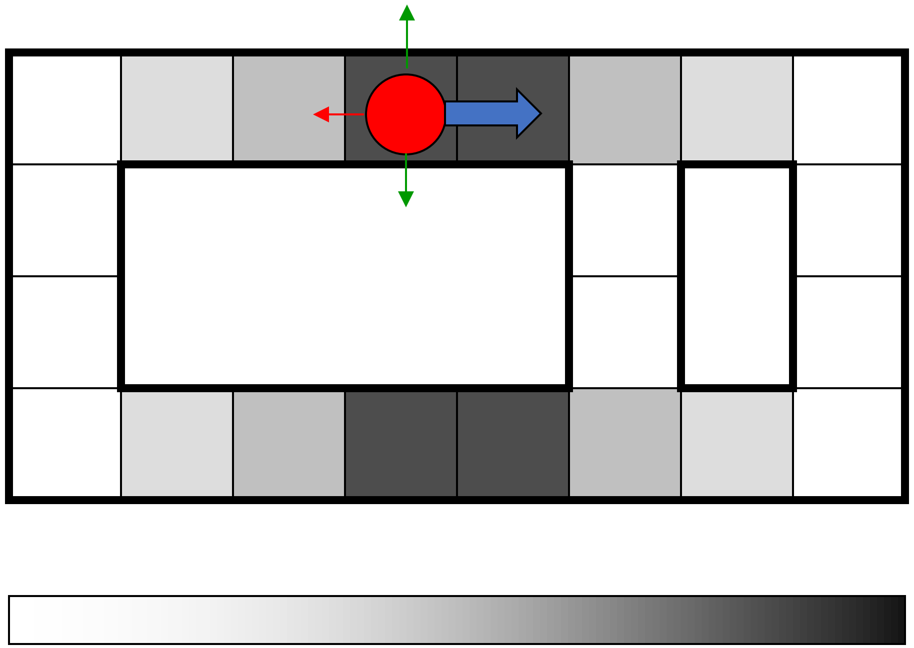




Prob 0 1

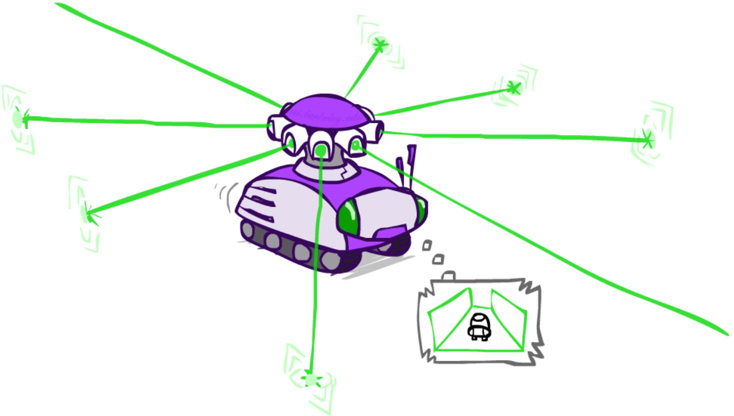
t=2

Example: Robot Localization

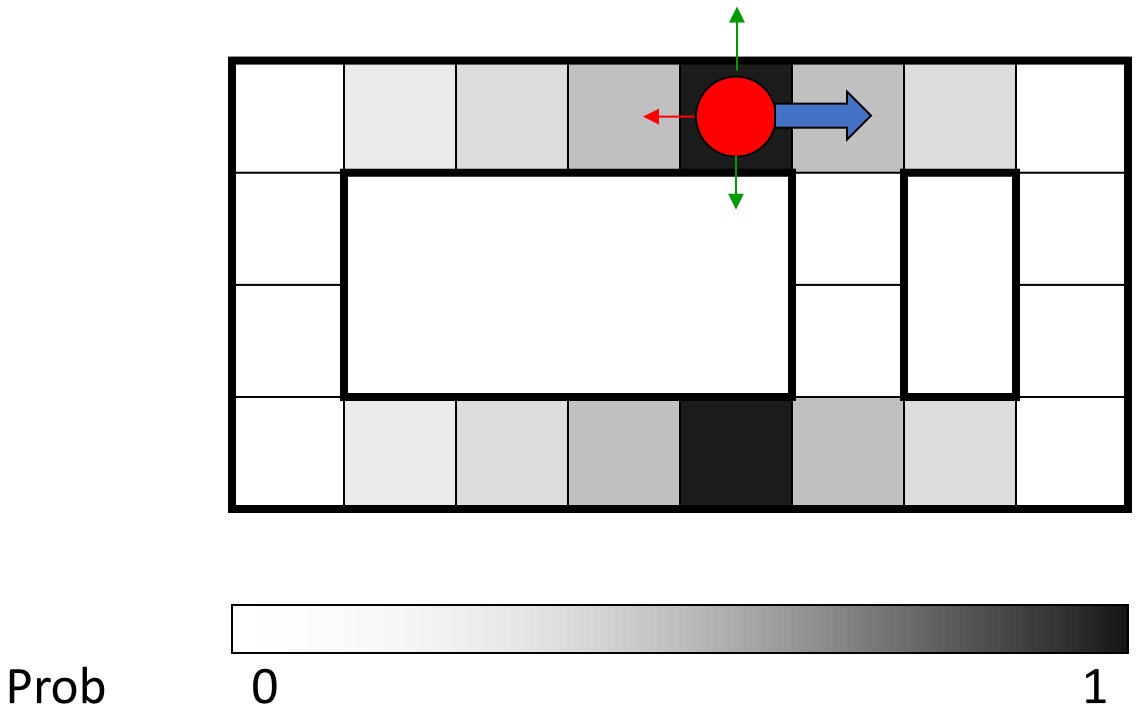


Prob 0 1

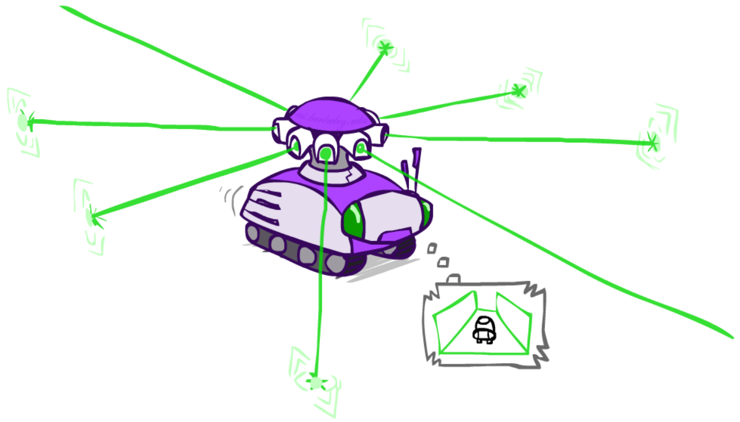
t=3



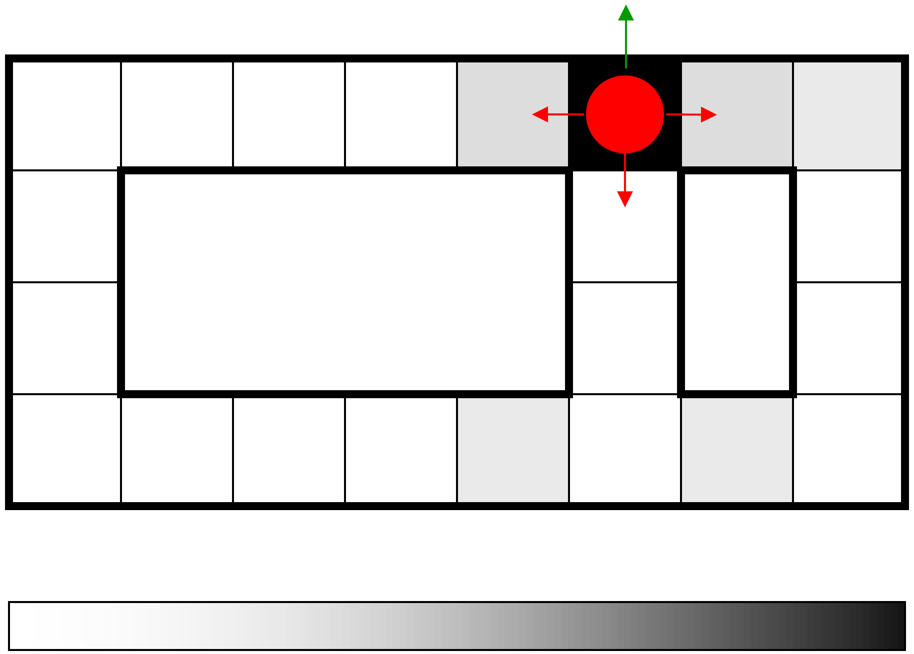
Example: Robot Localization



t=4

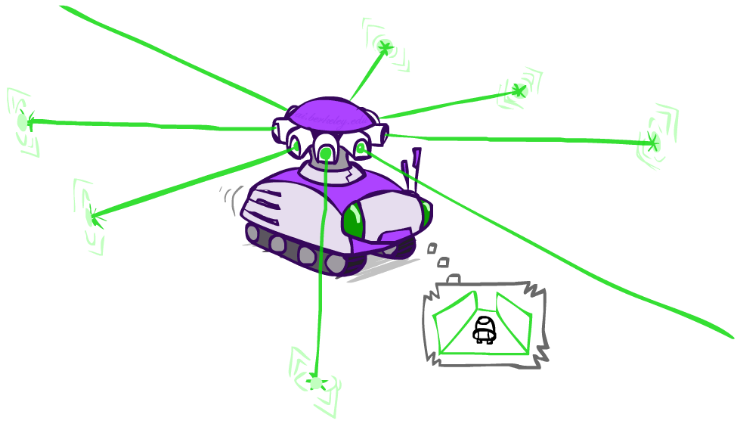


Example: Robot Localization



Prob 0 1

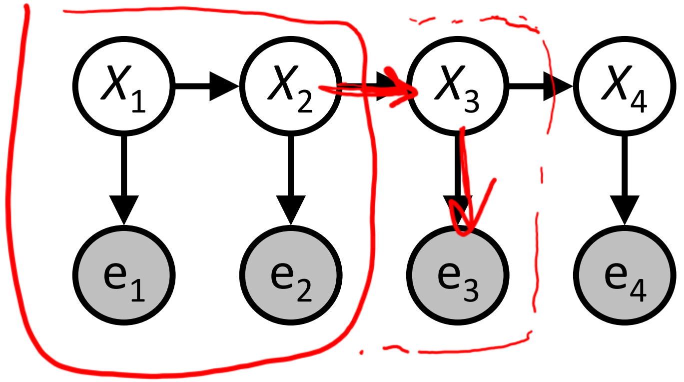
t=5



Exact Inference in HMM

Filtering

$P(X_1)$ $\quad P(X_t \mid X_{t-1})$ $\quad P(E_t \mid X_t)$



$$\boxed{P(X_t \mid e_{1:t}) = ?}$$

$$= \frac{P(X_1, e_1)}{P(e_1)}$$

Base case: $P(X_1 \mid e_1) \propto P(X_1, e_1) = P(X_1)P(e_1|X_1)$

Passage of time:

Suppose we have $P(X_t \mid e_{1:t})$.

How to calculate $P(X_{t+1} \mid e_{1:t+1})$?

$|X|$ $\quad |E|$

$$P(X_t \mid e_{1:t}) \longrightarrow P(X_{t+1}, X_t \mid e_{1:t}) \longrightarrow P(X_{t+1}, e_{t+1}, X_t \mid e_{1:t}) \longrightarrow P(X_{t+1}, e_{t+1} \mid e_{1:t}) \longrightarrow P(X_{t+1} \mid e_{1:t+1})$$

Joining $P(X_{t+1} \mid X_t)$ — Joining $P(e_{t+1} \mid X_{t+1})$ — Marginalize out X_t — Normalize

$$P(X_{t+1} \mid e_{1:t+1}) \propto \sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} \mid x_t)\, P(e_{t+1} \mid X_{t+1})$$

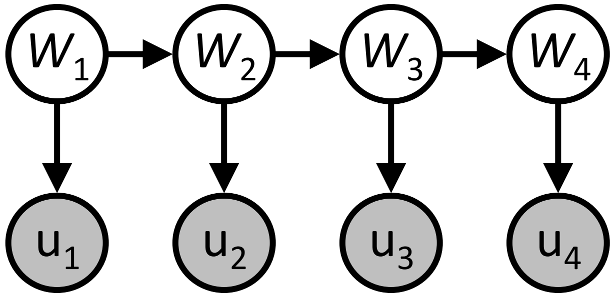
Time complexity?

$O(|X| \cdot |X|\, t)$

Exercise

$$P(W_1 \mid U_1 = T) \approx$$

W_1	$P(W_1 \mid U_1=T)$
s	2/11
r	9/11



$$P(W_2 \mid U_{1:2} = (T, F)\,) = ?$$

$$P(W_1 \mid U_1 = T) \propto P(W_1, U_1 = T) = P(W_1)\,P(U_1 = T \mid W_1)$$

$$= \begin{cases} W_1 = \text{sun}: & 0.5 \times 0.2 \\ W_1 = \text{rain}: & 0.5 \times 0.9 \end{cases}$$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	T	F
sun	0.2	0.8
rain	0.9	0.1

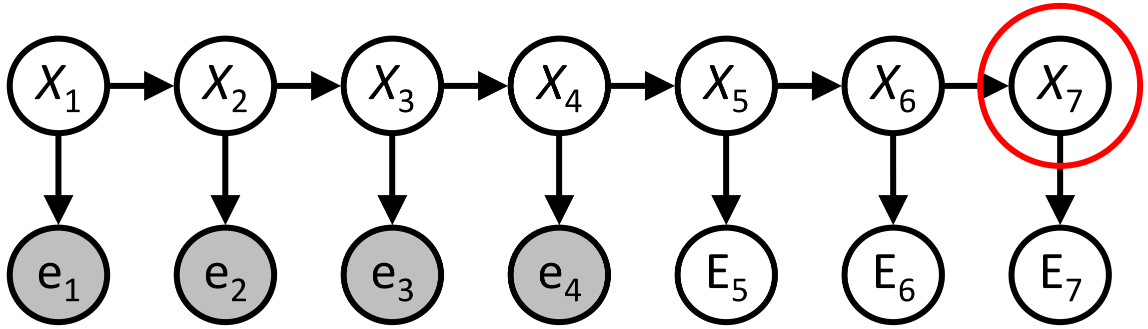
$$P(W_2 \mid U_1 = T, U_2 = F) \propto \sum_{w_1} P(w_1 \mid U_1 = T)\,P(W_2 \mid w_1)\,P(U_2 = F \mid W_2)$$

$$= P(W_1 = s \mid U_1 = T)\,P(W_2 \mid W_1 = s)\,P(U_2 = F \mid W_2)$$

$$+ P(W_1 = r \mid U_1 = T)\,P(W_2 \mid W_1 = r)\,P(U_2 = F \mid W_2)$$

W_1	$P(W_1)$
s	0.5
r	0.5

Prediction



$$P(X_{t+k} \mid e_{1:t}) = ?$$

We already have $P(X_t \mid e_{1:t})$ by filtering

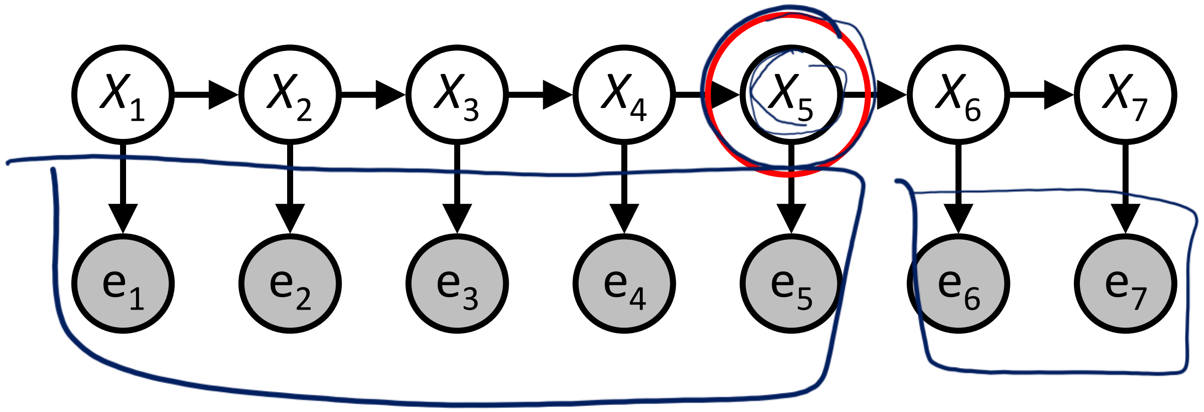
$$P(X_{t+1} \mid e_{1:t}) = \sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} \mid x_t)$$

$$P(X_{t+2} \mid e_{1:t}) = \sum_{x_{t+1}} P(x_{t+1} \mid e_{1:t}) P(X_{t+2} \mid x_{t+1})$$

$$\vdots$$

$$P(X_{t+k} \mid e_{1:t}) = \sum_{x_{t+k-1}} P(x_{t+k} \mid e_{1:t}) P(X_{t+k} \mid x_{t+k-1})$$

Smoothing



$P(X_k \mid e_{1:t}) = ?$ for some $k < t$

Here we introduce an approach slightly different from variable elimination.

$\times P(e_{k+1:t} \mid X_k, \cancel{e_{1:k}})$

$$P(X_k \mid e_{1:t}) \propto P(X_k, e_{k+1:t} \mid e_{1:k}) = \underbrace{P(X_k \mid e_{1:k})}_{\text{Forward algorithm (filtering)}} \; \underbrace{P(e_{k+1:t} \mid X_k)}_{\text{Backward algorithm}}$$

Just with one forward pass and one backward pass, we can calculate $P(X_k \mid e_{1:t})$ **for all k.**

$$\boxed{P(e_{k+1:t} \mid X_k)} \qquad k < t$$

Base Case : (k=t-1)

$$P(e_t \mid X_{t-1}) = \sum_{x_t} \overbrace{P(X_t \mid X_{t-1})\, P(e_t \mid X_t)}^{p(x_t, e_t \mid x_{t-1})}$$

Backward Pass: Given $P(e_{k+2:t} \mid X_{k+1})$

$$\begin{aligned}
P(e_{k+1:t} \mid X_k) &= \sum_{x_{k+1}} P(x_{k+1}, e_{k+1:t} \mid x_k) \\
&= \sum_{x_{k+1}} P(X_{k+1} \mid X_k)\, \underline{P(e_{k+1:t} \mid X_{k+1}, \cancel{X_k})} \\
&= \sum_{x_{k+1}} \underline{P(X_{k+1} \mid X_k)}\, \underline{P(e_{k+1} \mid X_{k+1})}\, \boxed{P(e_{k+2:t} \mid X_{k+1})}
\end{aligned}$$

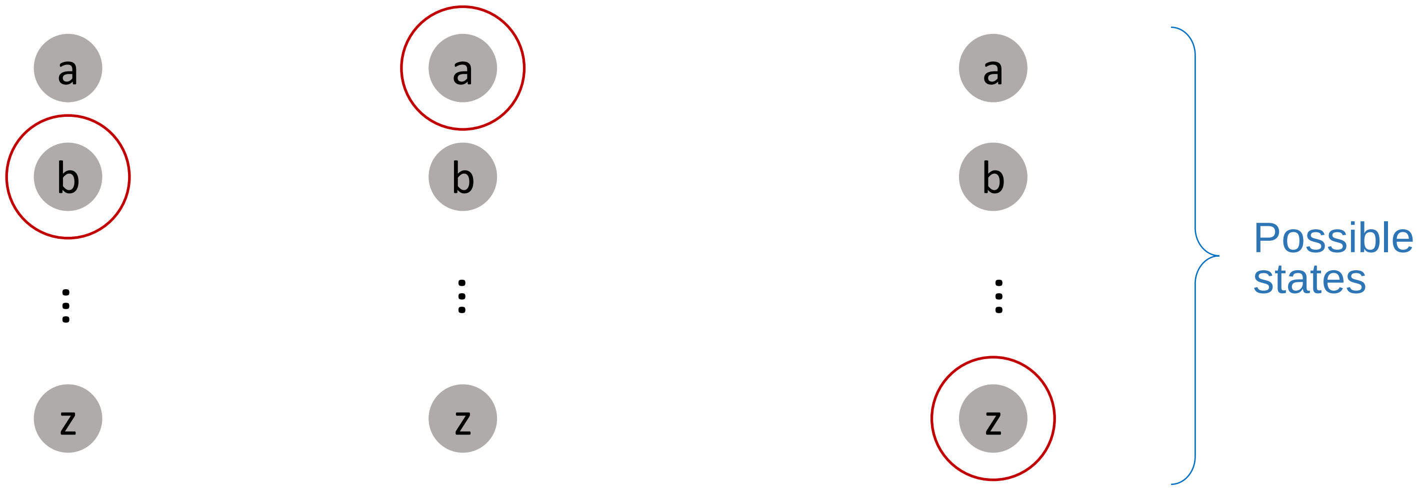
$$e_{k+1} \perp\!\!\!\perp e_{k+2:t} \mid X_{k+1}$$

Most-Likely Sequence

$$\boxed{\underset{X_{1:t}}{\operatorname{argmax}}\, P(X_{1:t} \mid e_{1:t}) = ?}$$

Find the sequence that maximizes the probability (e.g., speech recognition, sequence decoding)

$$P(X_{1:t} \mid e_{1:t}) \propto P(X_{1:t}, e_{1:t}) = \underbrace{P(X_1)P(e_1|X_1)}_{\text{Time 1}}\ \underbrace{P(X_2|X_1)P(e_2|X_2)}_{\text{Time 2}} \cdots \underbrace{P(X_t|X_{t-1})P(e_t|X_t)}_{\text{Time t}}$$

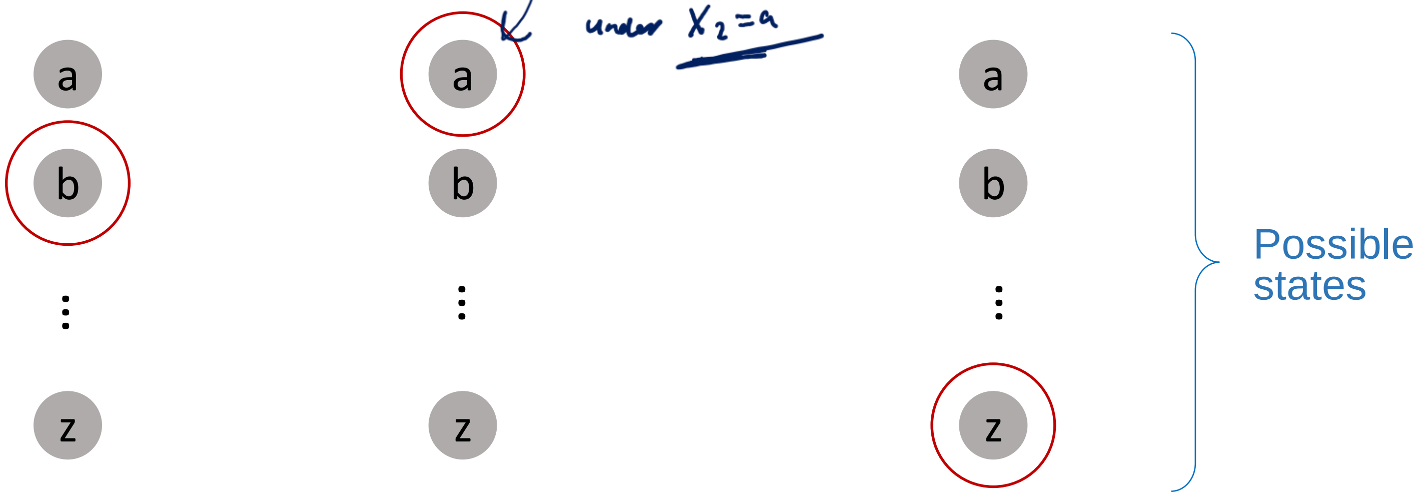


Find a sequence, e.g. $X_1 = b,\ X_2 = a,\ \ldots,\ X_t = z$ that maximize $P(X_{1:t}, e_{1:t})$

Most-Likely Sequence through Dynamic Programming

$$P(X_{1:t} \mid e_{1:t}) \propto P(X_{1:t}, e_{1:t}) = \underbrace{P(X_1)P(e_1|X_1)}_{\text{Time 1}}\ \underbrace{P(X_2|X_1)P(e_2|X_2)}_{\text{Time 2}} \cdots \underbrace{P(X_t|X_{t-1})P(e_t|X_t)}_{\text{Time t}}$$

maximum value under $X_2 = a$



Viterbi Algorithm

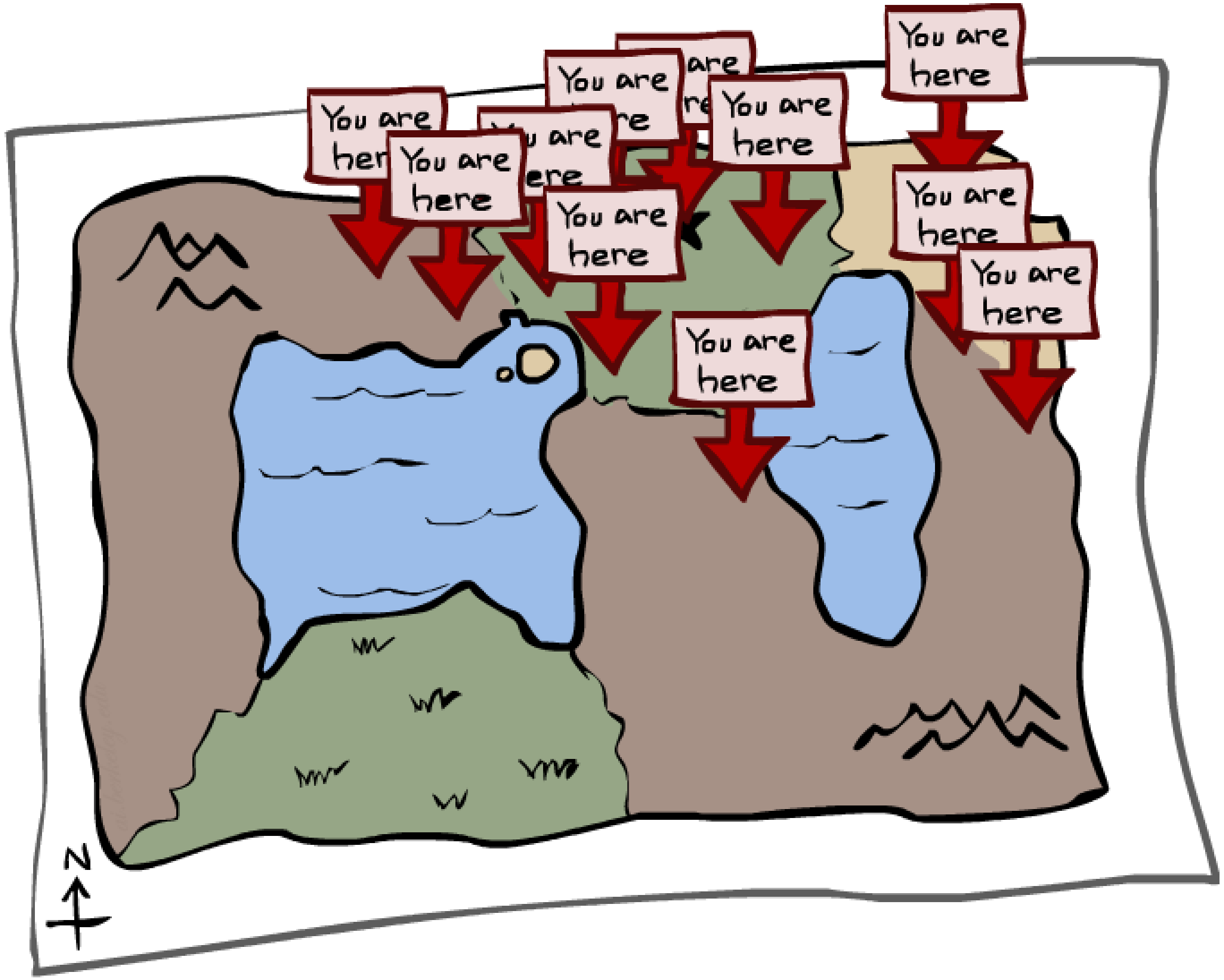
For each state s, let $\mathrm{Prob}[1][s] = P(X_1 = s)\ P(e_1|X_1 = s)$

For $k = 2, \ldots, t$:

For each states s, let $\mathrm{Prob}[k][s] = \max_{s'} \mathrm{Prob}[k-1][s'] \times P(X_k = s \mid X_{k-1} = s') \times P(e_k|X_k = s)$

Approximate Inference in HMM

Particle Filtering



Particle Filtering

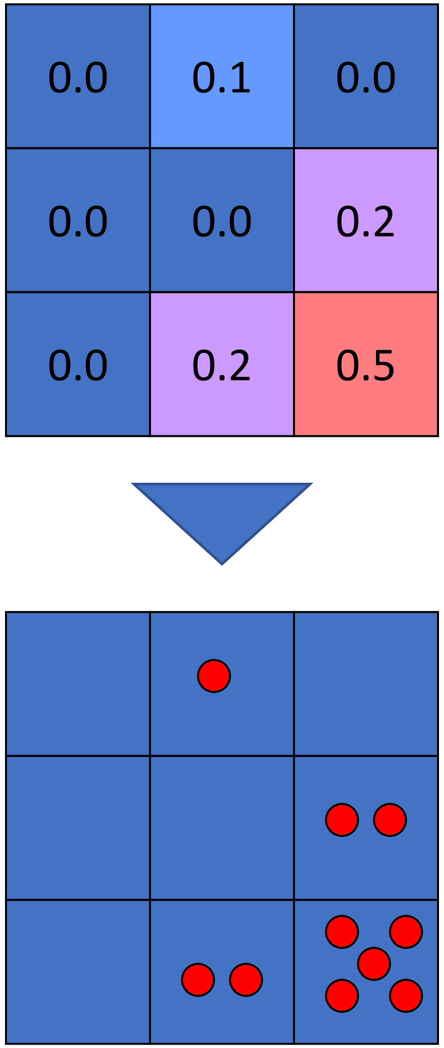
- Filtering: approximate solution

- Sometimes |X| is too big to use exact inference
  - |X| may be too big to even store P(X)
  - E.g. X is continuous

- Solution: approximate inference
  - Track samples of X, not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states

- This is how robot localization works in practice

- Particle is just new name for sample

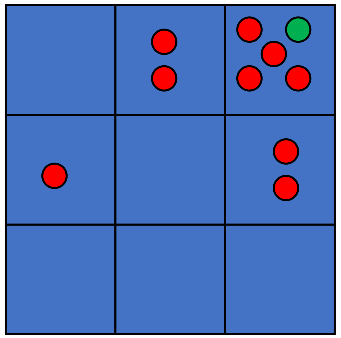


Representation: Particles

- Our representation of P(X) is now a list of N particles (samples)
  - Generally, N << |X|
  - Storing map from X to counts would defeat the point

- P(x) approximated by number of particles with value x
  - So, many x may have P(x) = 0
  - More particles, more accuracy

- For now, all particles have a weight of 1



Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

Particle Filtering: Elapse Time

- Each particle is moved by sampling its next position from the transition model

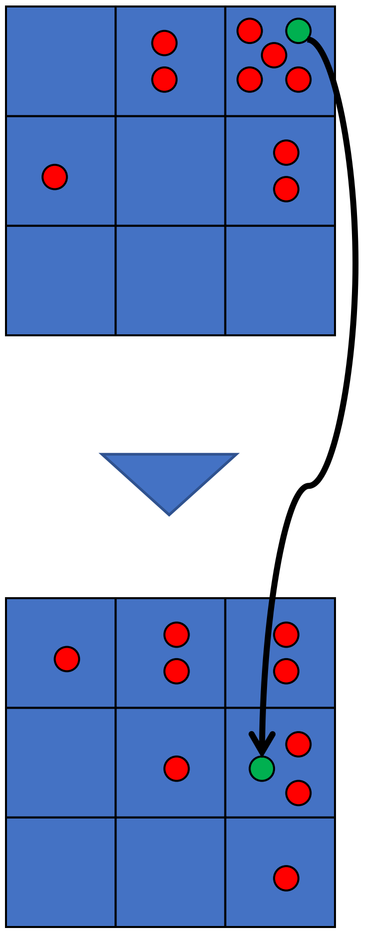
$$x' = \text{sample}(P(X'|x))$$

  - This is like prior sampling – samples' frequencies reflect the transition probabilities

- This captures the passage of time
  - If enough samples, close to exact values before and after (consistent)

Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)



Particle Filtering: Observe

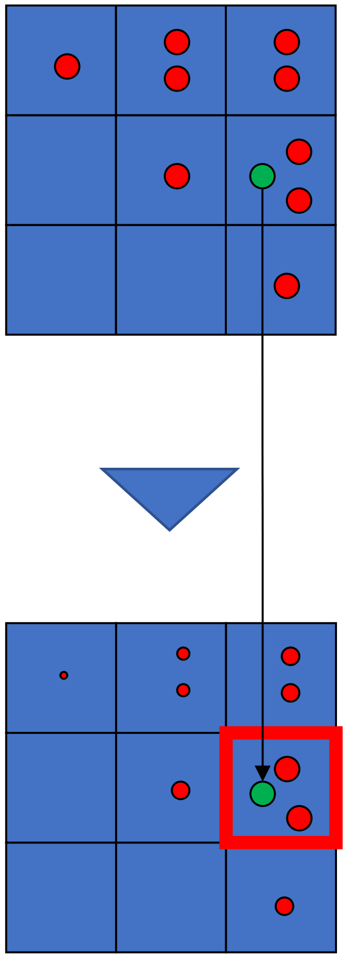
- Don’t sample observation, fix it

- Similar to **likelihood weighting**, downweight samples based on the evidence

$$w(x) = P(e|x)$$

- As before, the probabilities don’t sum to one, since all have been downweighted

Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

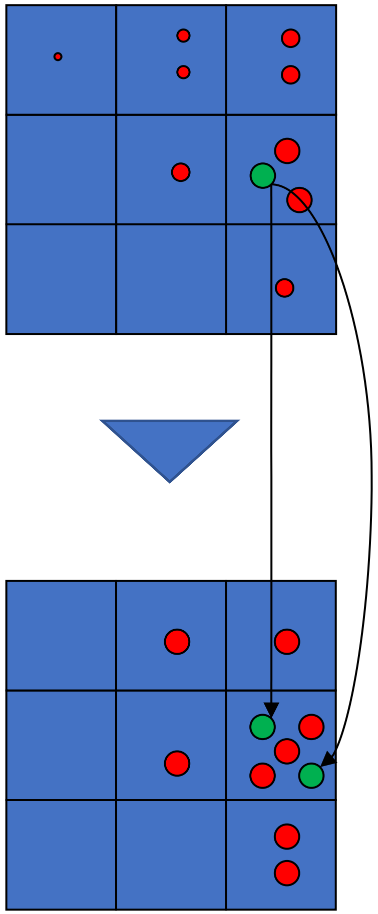


Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

Particle Filtering: Resample

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is similar to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

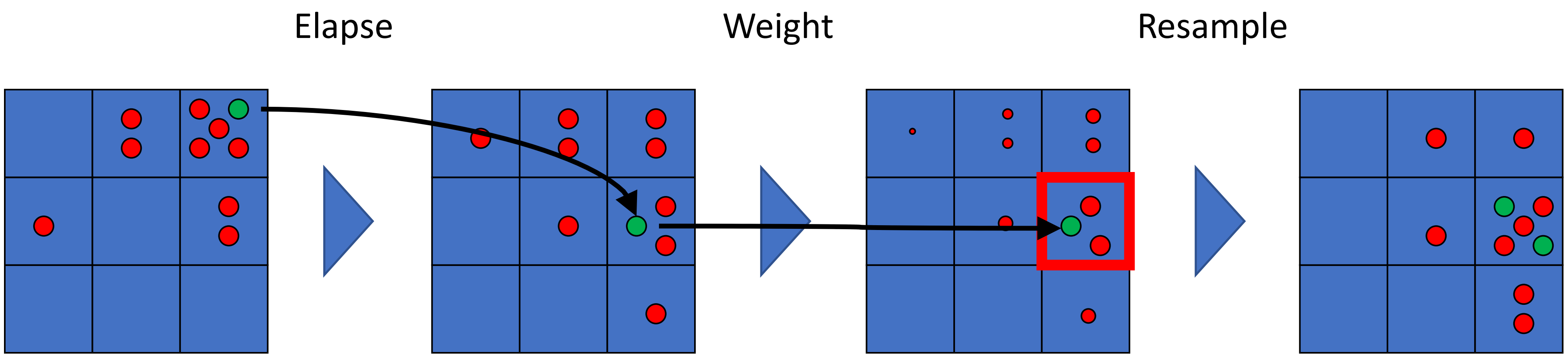


(New) Particles:
(3,2)
(2,2)
(3,2)
(2,3)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(3,2)

$\boxed{P(X_t | e_{1:t})}$

Recap: Particle Filtering

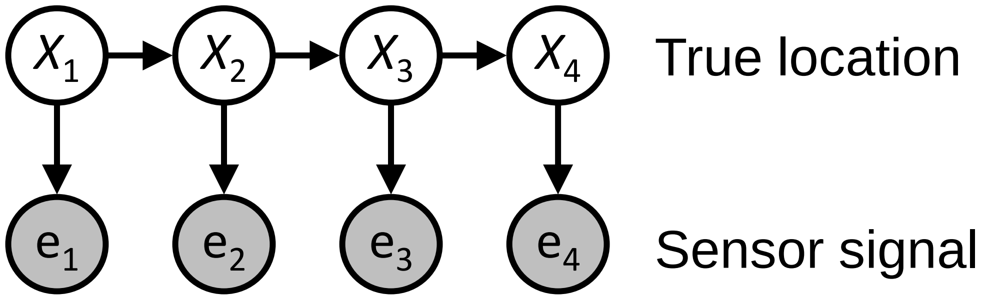
- Particles: track samples of states rather than an explicit distribution

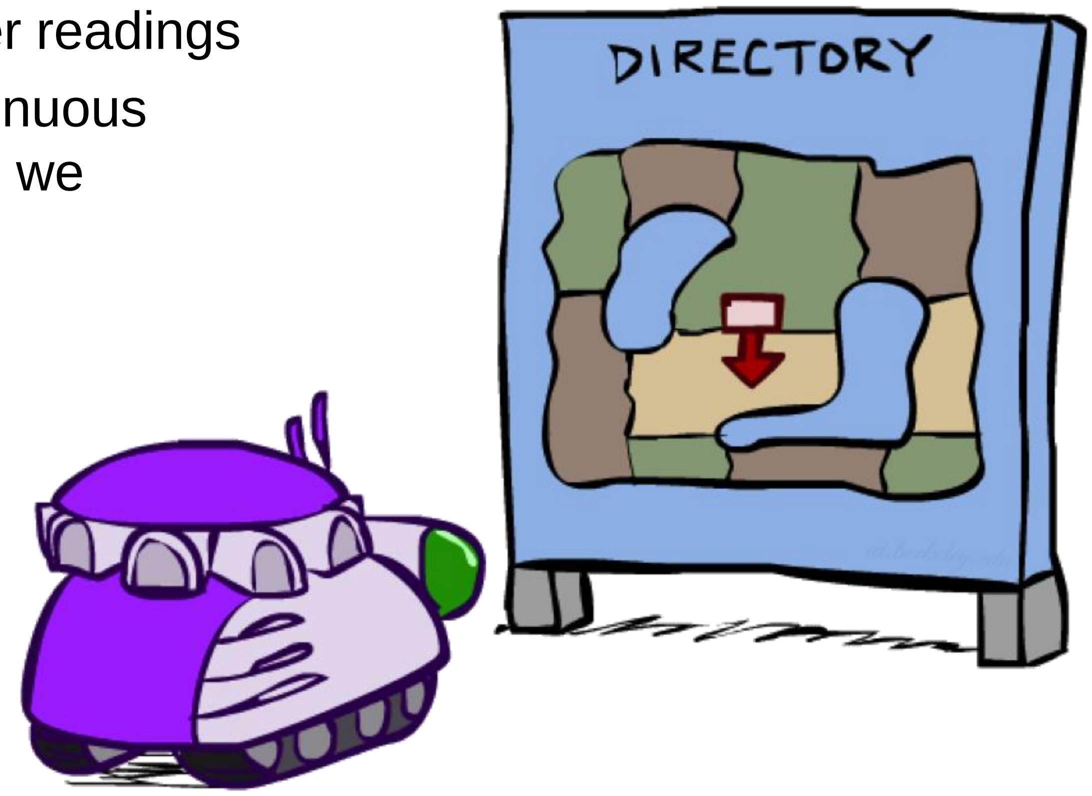


Particles:	Particles:	Particles:	(New) Particles:
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,3)	(2,3) w=.2	(2,2)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(3,2)	(3,1)	(3,1) w=.4	(2,3)
(3,3)	(3,3)	(3,3) w=.4	(3,3)
(3,2)	(3,2)	(3,2) w=.9	(3,2)
(1,2)	(1,3)	(1,3) w=.1	(1,3)
(3,3)	(2,3)	(2,3) w=.2	(2,3)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,2)	(2,2) w=.4	(3,2)

Robot Localization

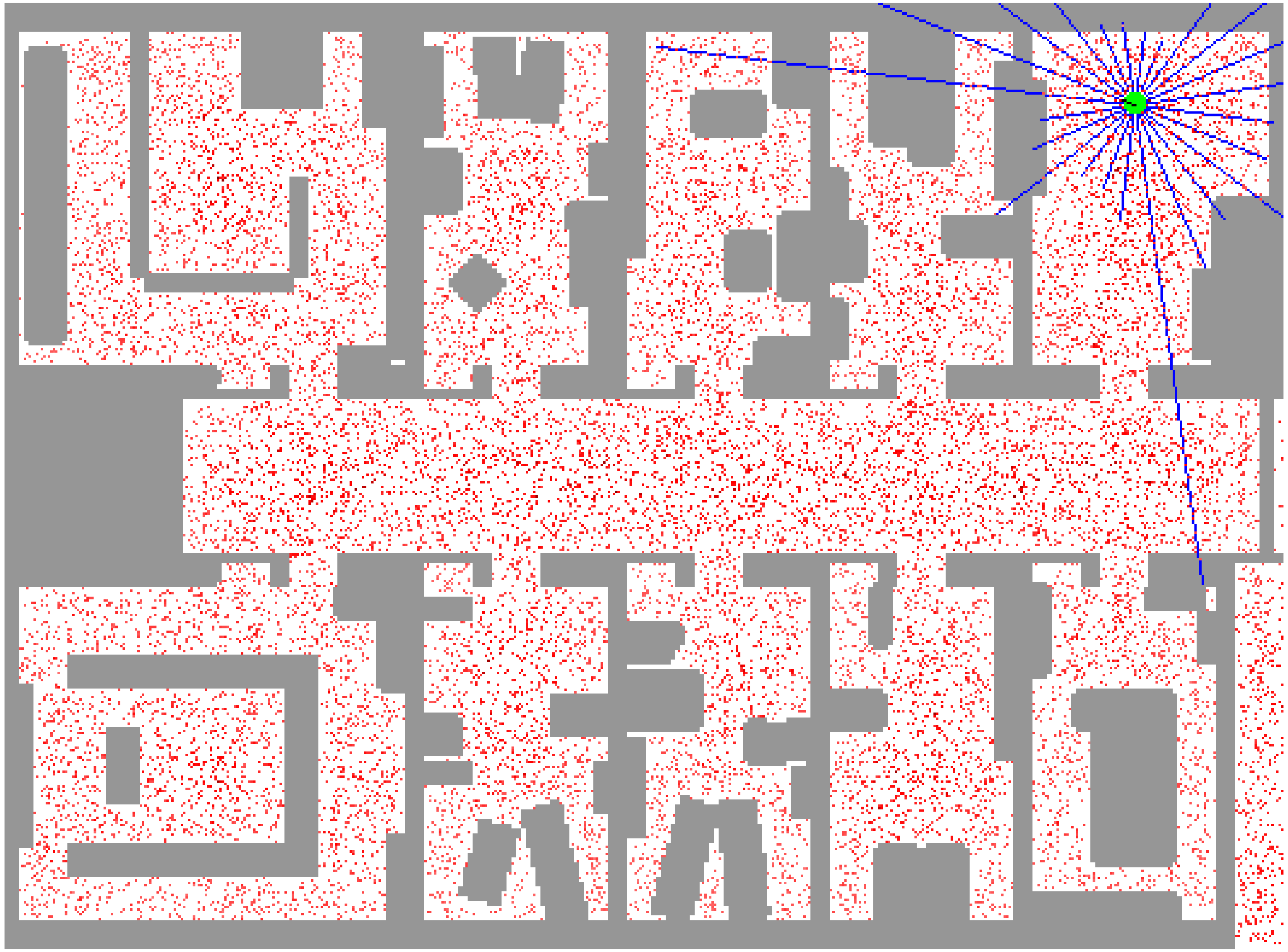
- In robot localization:
  - We know the map, but not the robot’s position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store B(X)
  - Particle filtering is a main technique





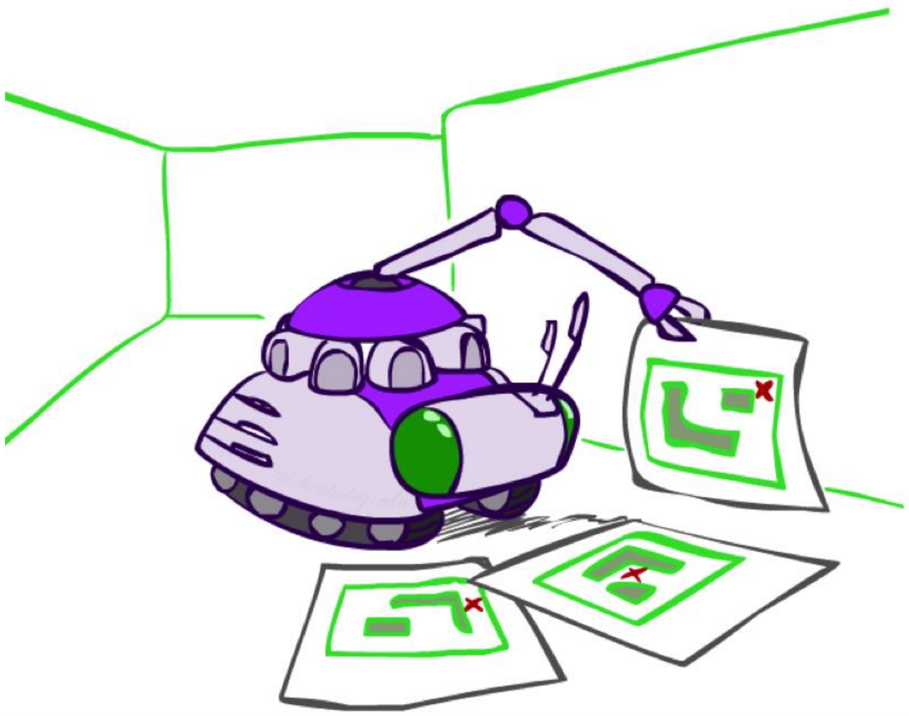
Particle Filter Localization (Sonar)

Particle Filter Localization (Laser)

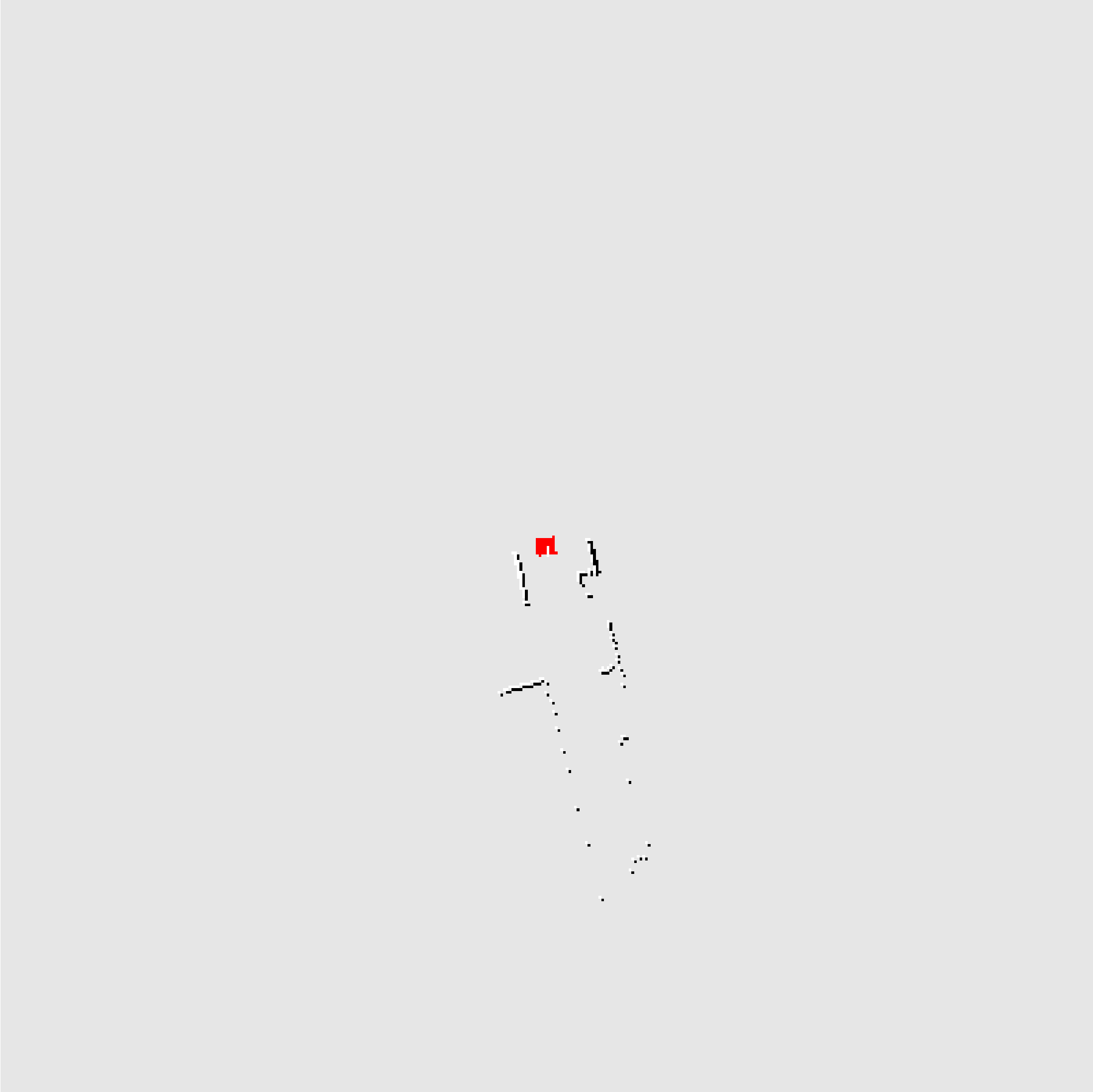


Robot Mapping

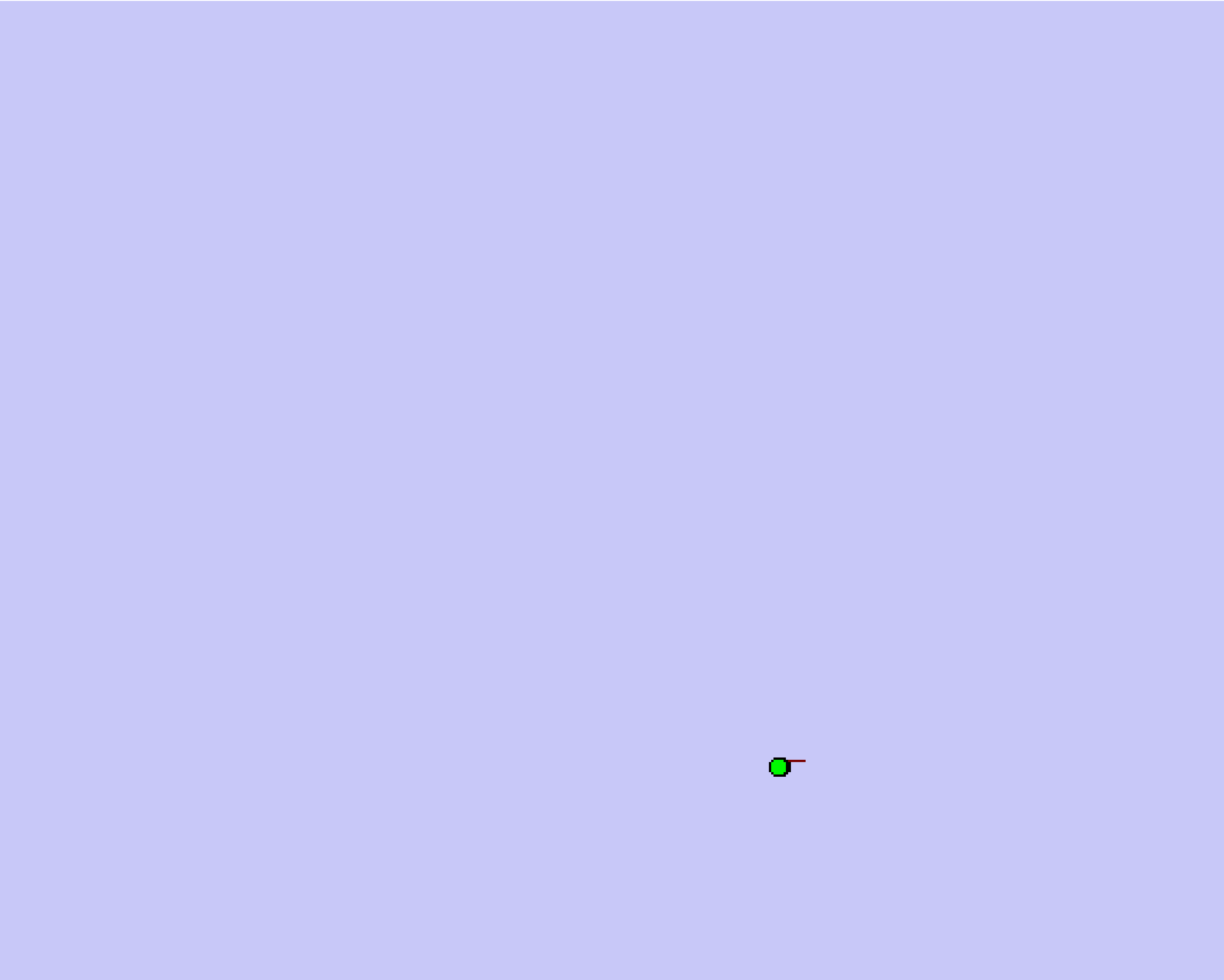
- SLAM: Simultaneous Localization And Mapping
  - We do not know the map or our location
  - State consists of position AND map!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods



Particle Filter SLAM – Video 1



Particle Filter SLAM – Video 2



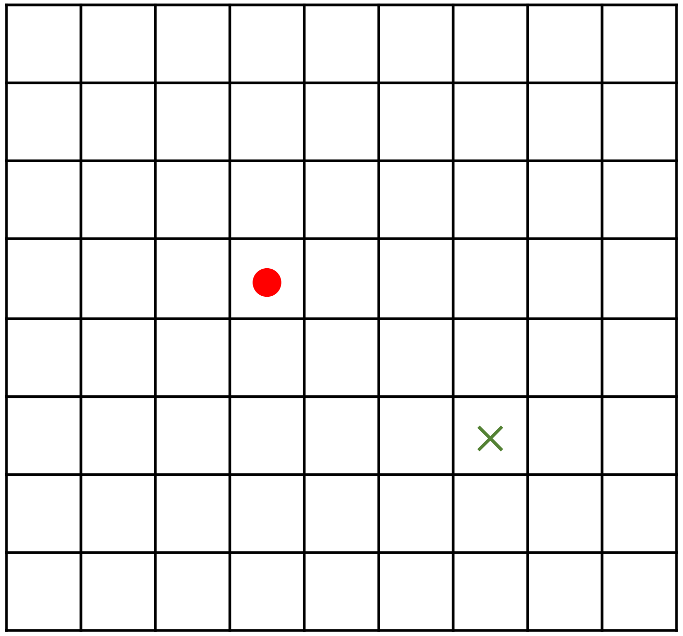
Particle Filtering

Localization: https://www.youtube.com/watch?v=NrzmH_yerBU&ab_channel=MATLAB

SLAM: https://www.youtube.com/watch?v=saVZtgPyyJQ&ab_channel=MATLAB

Some Failure Modes of Particle Filtering

Too few particles



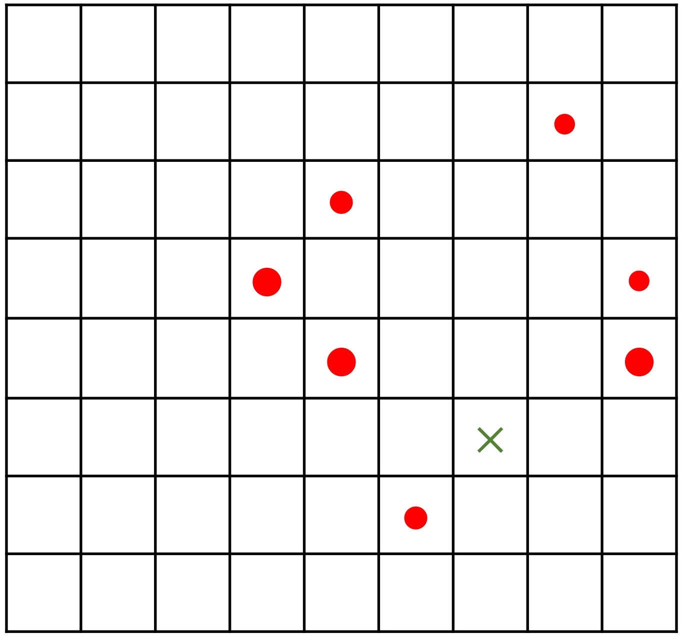
● Particle

× True location

→ The particle has to be dense enough to cover the true state

Some Failure Modes of Particle Filtering

Moderate number of particles but very static state transition



● Particle

× True location

Suppose every state always transitions to itself.

→ All particles and the true location will never move.

→ After several rounds of re-sampling, particles will accumulate to a single position.

Homework 4

Homework 4

1. Choice Questions (10 points)
   a. 10 questions.
   b. Answer directly on Gradescope
   c. The same requirements as the last time.

2. Program Questions (25 points)

   Ghostbusters and Bayes Nets

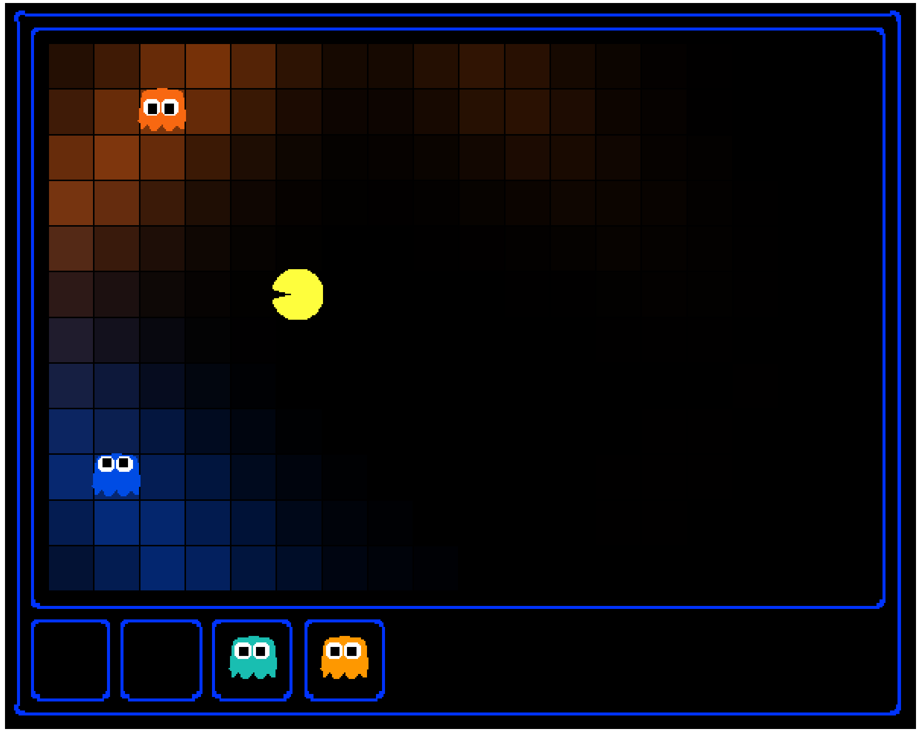
Introduction of Project 4: Ghostbusters and Bayes Nets

Color Blocks:

Indicate possible locations of each ghost based on distance readings.

Primary Task:

1. Implement inference to track ghost positions.
2. Improve on the default crude inference (shaded areas show possible ghost locations).
3. Use Bayes Nets for exact and approximate inference.

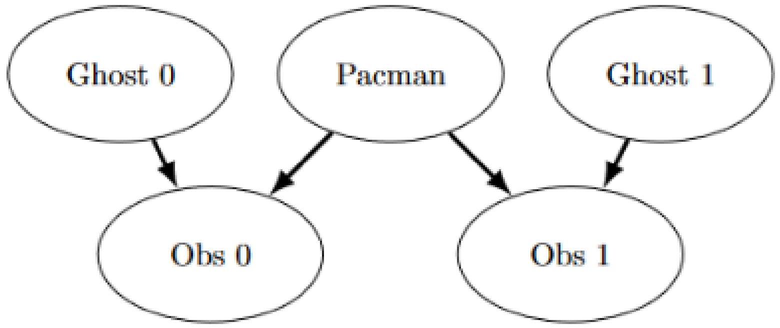


Question 1 (2 points): Bayes Net Structure

Objective:

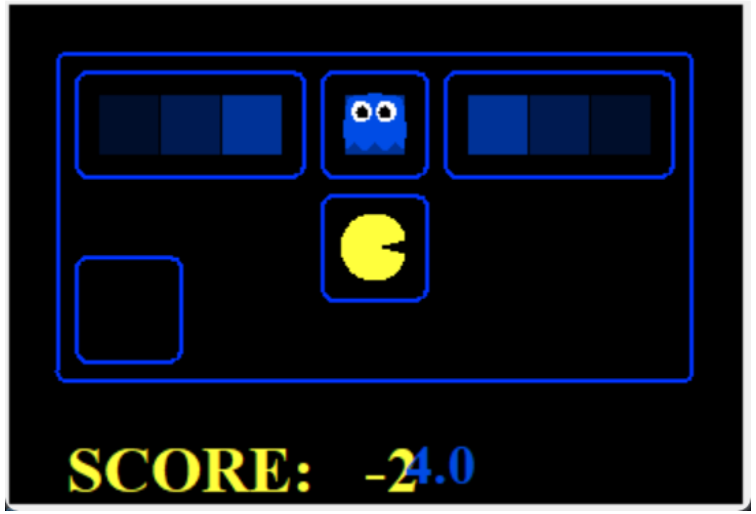
Implement *constructBayesNet* function in inference.py to create an empty Bayes Net structure as described.

Tasks:



1. Add variables and edges based on the diagram.
2. Pacman and the two ghosts can be anywhere in the grid
3. Observations here are non-negative, equal to Manhattan distances of Pacman to ghosts ± noise.

Question 2: Join Factors



Objective:

1. Takes a list of Factors and returns a new Factor.
2. The new Factor's entries are the product of corresponding rows of input Factors.

Assumptions:

joinFactors may operate on factors without probability tables (rows may not sum to 1).

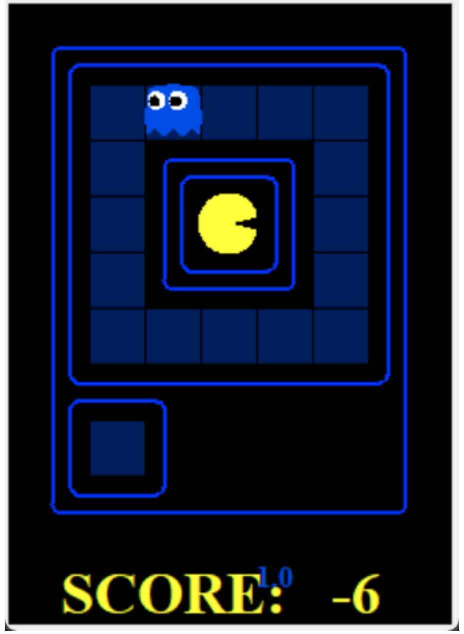
- `joinFactors` $(P(X \mid Y), P(Y)) = P(X, Y)$
- `joinFactors` $(P(V, W \mid X, Y, Z), P(X, Y \mid Z)) = P(V, W, X, Y \mid Z)$

Question 3: Eliminate (not ghosts yet)

Objective:

1. Takes a Factor and a variable to eliminate.
2. Returns a new Factor without that variable, by summing entries differing in the eliminated variable's value.

- Examples:
  - $eliminate(P(X,Y|Z),Y) = P(X|Z)$
  - $eliminate(P(X,Y|Z),X) = P(Y|Z)$

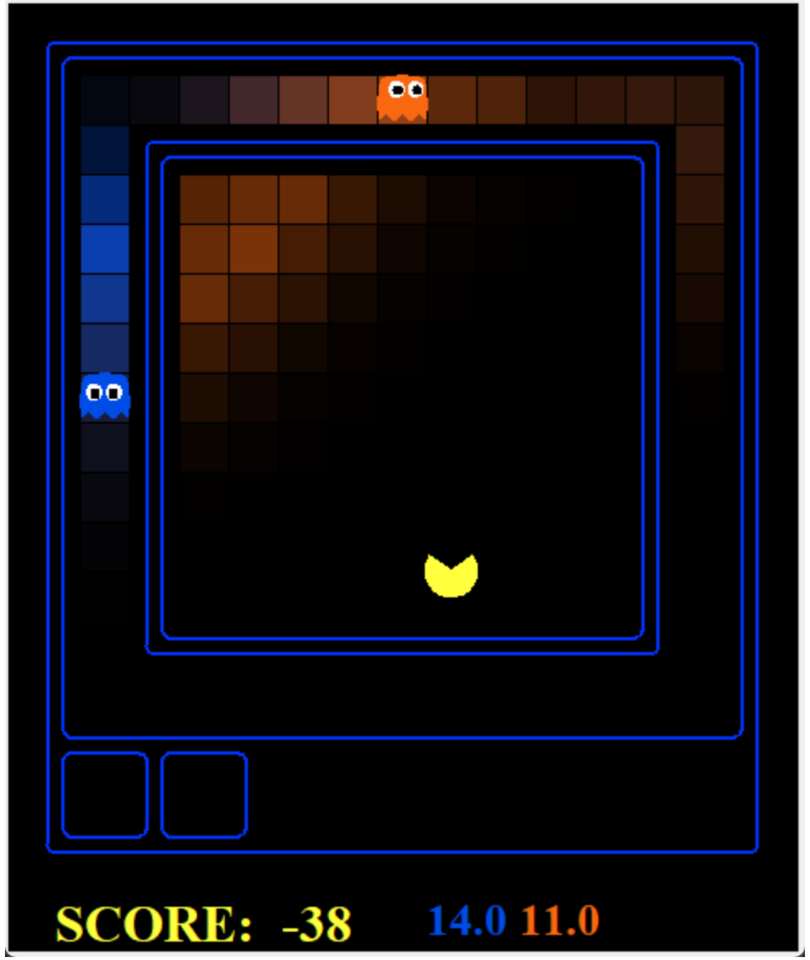


Question 4: Variable Elimination

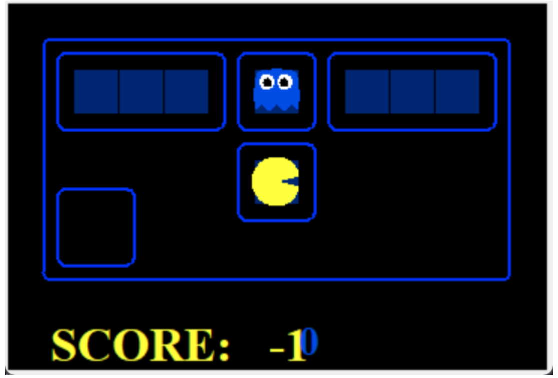
Objective: Answers a probabilistic query represented using, A BayesNet, A list of query variables and Evidence.

Hints and Observations:

1. Refer to *inferenceByEnumeration* function for guidance.
2. Sum of probabilities should equal 1 (to ensure it's a true conditional probability).
3. Enumeration joins all variables first and then eliminates all hidden variables.
4. Variable Elimination interleaves join and eliminate, processing one hidden variable at a time.
5. Handle cases where a factor has only one unconditioned variable after joining.



Question 5a and 5b



5a objective:

Complete *DiscreteDistribution* to extends the Python dictionary, where keys are elements of the distribution, and values are the associated weights.

5b objective:

Complete *getObservationProb* to Calculates the probability of a noisy distance reading between Pacman and a ghost.

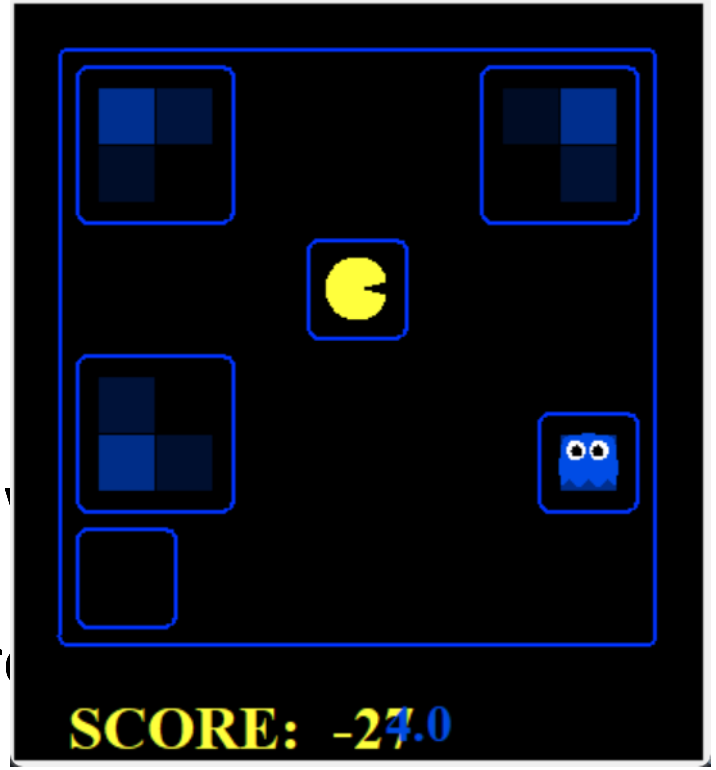
Question 6: Exact Inference Observation

Objective:

Implement *observeUpdate* to update the belief distribution over ghost positions based on Pacman's sensor observations.

Display Behavior:

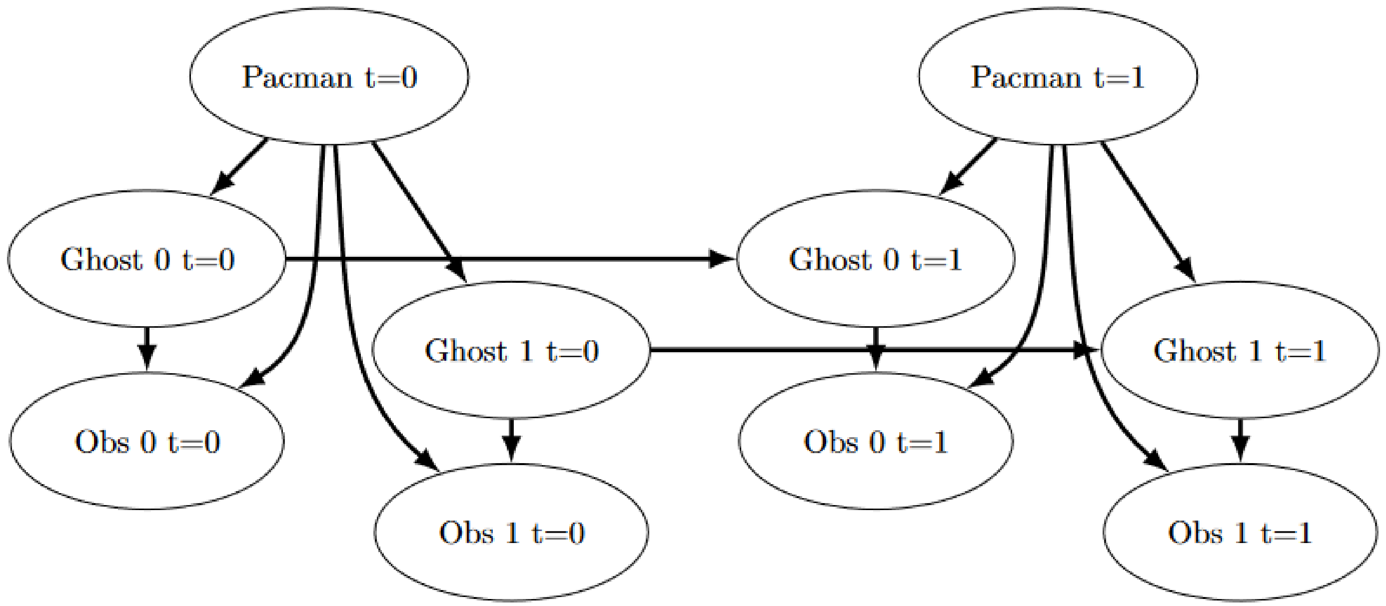
- High posterior beliefs are shown as bright colors; lo
  beliefs are dim.
- Beliefs should start broad and narrow down as mor
  evidence is collected.



Question 7: Exact Inference with Time Elapse

Objective:

Implement the *elapseTime* to update ghost position beliefs over time based on movement patterns without observing them.



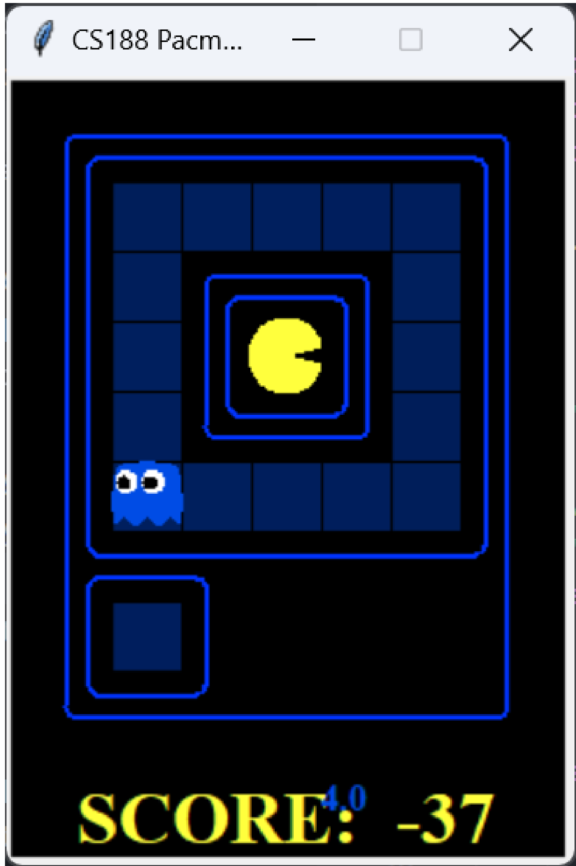
Question 7: Exact Inference with Time Elapse

Notes:

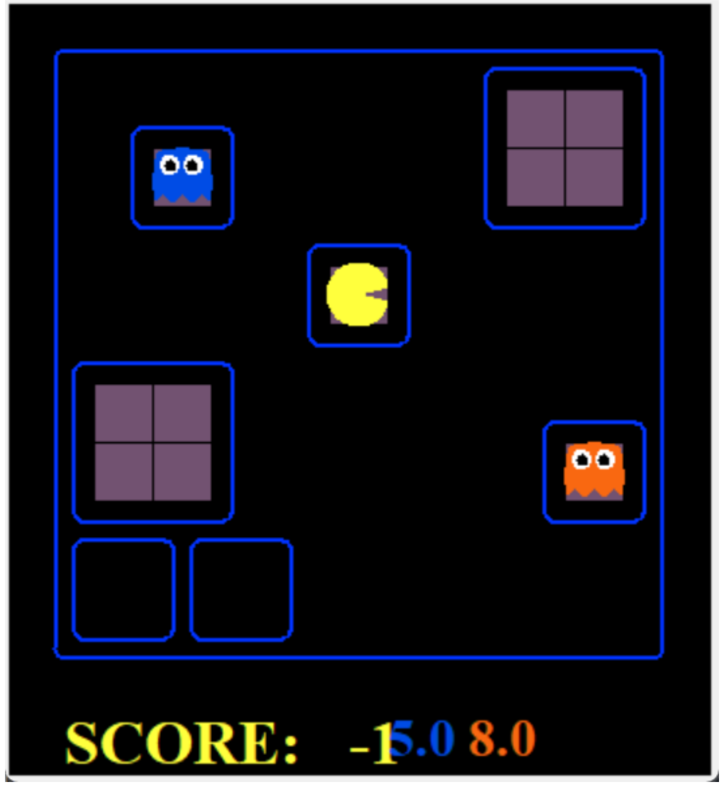
- If code is slow, reduce calls to *self.getPositionDistribution*.
- Pacman's belief distribution adjusts based on possible ghost movements without direct observation.
- Beliefs will adapt to the board geometry and likely ghost moves over time.

Special Ghost Behavior:

- GoSouthGhost: A ghost that tends to move south over time.
- Pacman's belief distribution should focus near the board's bottom as the GoSouthGhost moves south.



Question 8: Exact Inference Full Test

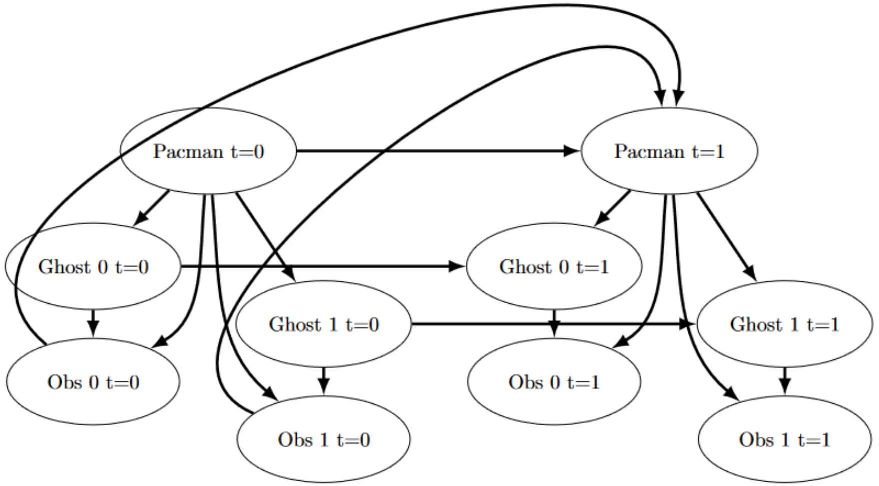


Objective:

1. The agent should select actions based on the belief distribution to move towards the closest ghost.

Tasks:

1. Identify the most likely position of each uncaptured ghost.
2. Choose an action that minimizes the maze distance to the closest ghost.



Question 9: Approximate Inference Initialization & Beliefs

Objective:

Implement *initializeUniformly* and *getBeliefDistribution* to set up a particle filtering algorithm to track a single ghost.

Method Details:

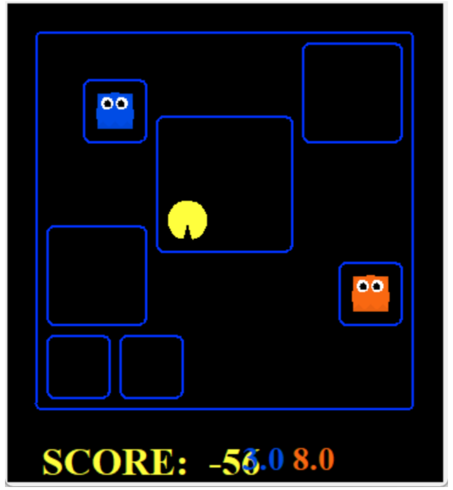
1. *initializeUniformly*:

Distribute particles evenly across all legal ghost positions (ensures a uniform prior).

Consider using the mod operator to achieve even distribution.

1. *getBeliefDistribution*:

Convert the list of particles into a *DiscreteDistribution* object representing the belief distribution.



Question 10 & 11: Approximate Inference Observation

Q10: Approximate Inference Observation

Implement the *observeUpdate* for updating the weight distribution over self.particles based on Pacman’s observation.

Q11: Approximate Inference with Time Elapse

Implement the elapseTime to update *self.particles* by constructing a new list of particles that corresponds to each existing particle advancing a time step.