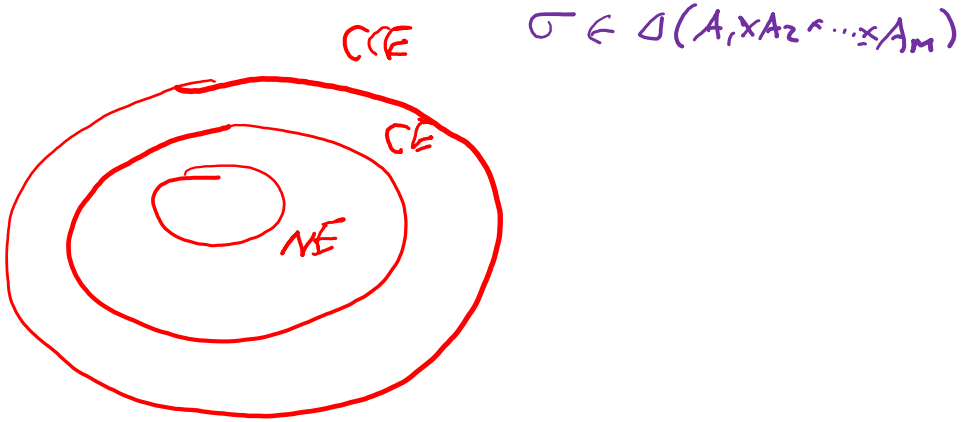
Price of Anarchy and Smooth Games

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How Good Are Equilibria?



- What do learning dynamics converge to?
  - No-regret learning → CCE
  - No-swap-regret learning → CE
- How good are these independent players' outcomes as a whole compared with the outcome with a central planner?
- **Price of anarchy (PoA):** how much is lost because players are selfish
- **Smooth game:** a recipe for bounding PoA

Price of Anarchy

Price of Anarchy

Cost-minimization game:

$$\text{PoA} = \frac{\max_{\sigma \in \text{Eq}} C(\sigma)}{\min_{a} C(a)} \qquad C(a) = \sum_j \; c_j(a)$$ (sum of costs over players)

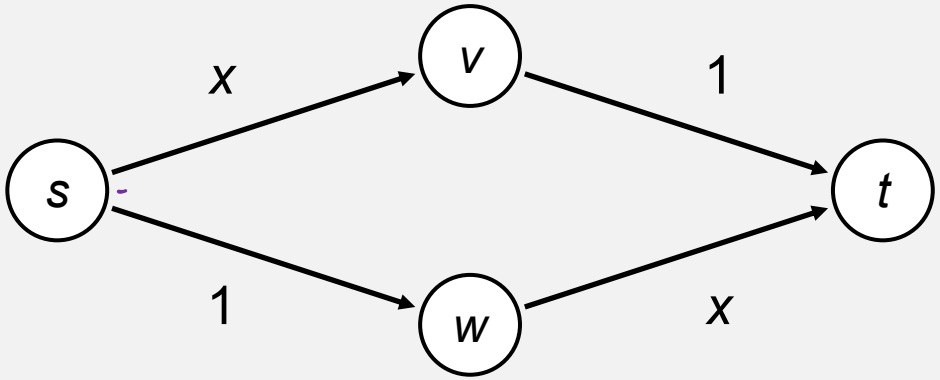
Utility-maximization game:

$$\text{PoA} = \frac{\max_{a} U(a)}{\min_{\sigma \in \text{Eq}} U(\sigma)} \qquad U(a) = \sum_j \; u_j(a)$$

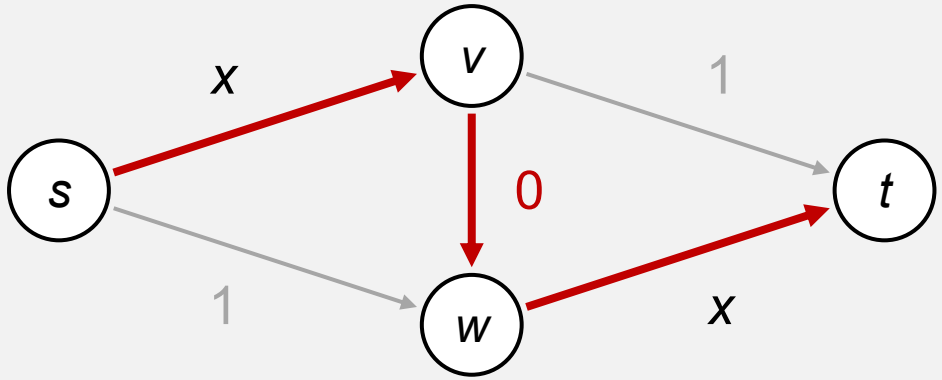
Recall: Braess's paradox

1 unit of traffic goes from s to t, and every driver takes a fastest route.
Edge labels are travel times, where x = fraction of drivers using that edge.

Before



After: add a road v → w with travel time 0



$\max_{e_f} C(e_f) = 2$

$\min_{a} C(a) = 1.5$

$PoA = \frac{2}{1.5} = \frac{4}{3} \approx 1.33$

Smooth Games

Smooth Games (cost minimization)

Definition. A cost-minimization game is (λ, μ)**-smooth** for $\lambda \geq 0$ and $\mu < 1$ if for there exists a socially optimal action profile $a^\star$ such that

$$\sum_j c_j(a_j^\star, a_{-j}) \leq \lambda C(a^\star) + \mu C(a)$$

for any action profile a.

$\Rightarrow \mathbb{E}_{a\sim\sigma}[C(a)] \leq \frac{\lambda}{1-\mu} C(a^*)$

Let σ be a CCE. By smooth game def

$$\mathbb{E}_{a\sim\sigma}[C(a)] = \mathbb{E}_{a\sim\sigma}\left[\sum_j c_j(a)\right] \underset{\text{CCE}}{\leq} \mathbb{E}_{a\sim\sigma}\left[\sum_j c_j(a_j^*, a_{-j})\right] \leq \lambda C(a^*) + \mu \mathbb{E}_{a\sim\sigma}[C(a)]$$

Smooth Game

$PoA(NE) \leq PoA(CE) \leq PoA(CCE)$

$NE \subseteq CE \subseteq CCE$

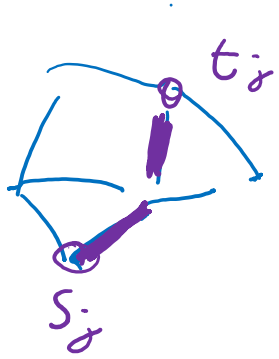
Theorem. For any (λ, μ)-smooth cost-minimization game,

$PoA(NE) \leq PoA(CE) \leq \text{PoA(CCE)} \leq \frac{\lambda}{1-\mu}$

Corollary. Suppose every player uses a **no-regret algorithm** to repeatedly play a (λ, μ)-smooth cost-minimization game. Then

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{j} c_j(a_t) \leq \frac{\lambda}{1-\mu}\min_{a}\sum_{j} c_j(a) + o(1)$$

Example: Atomic Congestion Game



Atomic Congestion Game

- Resources E (e.g., edges of a network)
- Player j chooses a set of resources $a_j \in \mathcal{A}_j \subseteq 2^E$ (e.g., a path from s_j to t_j)
- Load of resource e: $n_e(a) = \left|\{j:\ e \in a_j\}\right|$
- Cost of player j: $c_j(a) = \sum_{e \in a_j} \ell_e(n_e(a))$ where ℓ_e is nondecreasing
- Social cost: $C(\mathrm{a}) = \sum_j c_j(a)$

Atomic Congestion Game with Affine Cost is Smooth

Theorem. An atomic congestion game with cost function $\ell_e(n) = \alpha_e n + \beta_e$ for $\alpha_e, \beta_e \geq 0$ is $\left(\frac{5}{3}, \frac{1}{3}\right)$-smooth.

Theorem. For any atomic congestion game,

$$\mathrm{PoA}(\mathrm{CCE}) \leq \frac{5}{2}.$$

$$\frac{\lambda}{1-\mu} = \frac{5/3}{1-1/3} = \frac{5}{2}$$

Furthermore, there exists an atomic congestion game with $\mathrm{PoA}(\mathrm{NE}) = \frac{5}{2}$.

$\forall\, a', a$

$$\sum_j c_j(a'_j, a_{-j}) \le \lambda \sum_j c_j(a') + \mu \sum_j c_j(a) = \sum_j \sum_{e \in a_j} (\alpha_e n_e + \beta_e) = \sum_{e \in E} n_e(\alpha_e n_e + \beta_e) = \sum_{e \in E} (\alpha_e n_e^2 + \beta_e n_e)$$

$$= \sum_{e \in E} (\alpha_e n_e'^2 + \beta_e n_e')$$

$n_e = \sum_j \mathbb{1}\{e \in a_j\}$

$n_e' = \sum_j \mathbb{1}\{e \in a_j'\}$

$$\text{LHS} \le \sum_j \sum_{e \in a'_j} (\alpha_e(n_e + 1) + \beta_e) \le \sum_{e \in E} n'_e(\alpha_e(n_e + 1) + \beta_e) = \alpha_e n'_e(n_e + 1) + n'_e \beta_e$$

The theorem can be proven if we can prove $n'_e(n_e + 1) \le \frac{5}{3} n_e'^2 + \frac{1}{3} n_e^2$ and $n'_e \le \frac{5}{3} n'_e + \frac{1}{3} n_e$

They indeed hold for any non-negative integers n'_e, n_e.

Non-Atomic Congestion Game

Non-atomic congestion game $=$

Atomic congestion game with large population (each player controls negligible traffic)

Theorem. A non-atomic congestion game with cost function $\ell_e(n) = \alpha_e n + \beta_e$ for $\alpha_e, \beta_e \geq 0$ is $\left(1, \frac{1}{4}\right)$-smooth.

Theorem. For any non-atomic congestion game,

$$\mathrm{PoA}(\mathrm{CCE}) \leq \frac{4}{3}.$$

Furthermore, there exists an atomic congestion game with $\mathrm{PoA}(\mathrm{NE}) = \frac{4}{3}$.

Smooth Games (utility maximization)

Definition. A utility-maximization game is $(\boldsymbol{\lambda}, \boldsymbol{\mu})$**-smooth** for $\lambda, \mu \geq 0$ if there exists a socially optimal action profile $a^\star$ such that

$$\sum_j u_j(a_j^\star, a_{-j}) \geq \lambda U(a^\star) - \mu U(a)$$

for any action profile a.

(λ, μ)-smooth $\Rightarrow$ $PoA(CCE) \leq \frac{\lambda}{1+\mu}$

Example: First-Price Auction

$U(b) = \sum_j u_j(b) + \text{Rev(seller)}$, b_{winner}

- One item, M bidders; bidder j has value v_j and submits a sealed bid b_j
- The highest bid wins and pays its own bid
- Utility: $u_j(b) = v_j - b_j$ if j wins, and 0 otherwise

$u_{winner} - b_{winner} = u_{winner}$

Bidder	Value	Bid	Wins?	Pays	Utility
1	10	7	yes	7	10 − 7 = 3
2	6	5	no	0	0
3	4	3	no	0	0

Seller's revenue: 7. Welfare = value of the winner = 10 = 3 + 7

- Not truthful: bidding $b_j = v_j$ gives utility 0, so bidders bid below their values
- $U(b)$ = value of the winner; $\max_b U(b) = \max_j v_j$

Example: First-Price Auction

Theorem. First-price auction is $(1 - 1/e, 0)$-smooth.

Theorem. For any first-price auction, let σ be any CCE, then

$$\mathbb{E}_{b\sim\sigma}[U(b)] \geq (1 - 1/e)\max_{b} U(b)$$