Full-Information Online Learning with Adversarial Reward

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The Expert Problem

Alternative protocol:

Environment **decides** the reward vector r_t (not revealing)

Learner chooses an expert a_t

Environment reveals r_t

Given: set of experts $\mathcal{A} = \{1, \dots, A\}$

For time $t = 1, 2, \dots, T$:

Learner chooses a distribution over experts $p_t \in \Delta_{\mathcal{A}}$

Environment **decides** and reveals the reward vector $r_t = (r_t(1), \dots, r_t(A))$

Adversarial environment: $r_1(a), \dots, r_T(a)$ do not have the same mean

$$\mathrm{Regret} = \max_{a \in \mathcal{A}} \sum_{t=1}^{T} r_t(a) - \sum_{t=1}^{T} \langle p_t, r_t \rangle$$

Strategies?

- Follow the leader

$$a_t = \max_{a \in \mathcal{A}} \left\{ \sum_{i=1}^{t-1} r_i(a) \right\}$$

time	1	2	3	4	5 . .
action 1	1/2	1	0	1	0 . . . .
action 2	1	0	1	0	1 . . . .

Learner total reward ≤ 1

Total reward of best action

$\approx \frac{T}{2}$

Incremental Updates

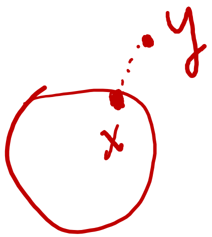
Projected gradient ascent:

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

Exponential weight updates:

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))}$$

learner's expected reward (p) = $\langle p, r_t \rangle$



$$\Pi_\Delta(y) = \underset{x \in \Delta}{\operatorname{argmin}} \|x - y\|_2$$

Equivalent Forms of EWU

$$p_{t+1}(a) \propto \underline{\underline{p_t(a)}} \exp(\eta r_t(a)) \propto p_{t-1}(a) \exp(\eta r_{t-1}(a)) \exp(\eta r_t(a)) \cdots$$

$$p_{t+1}(a) = \frac{p_t(a) \exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a') \exp(\eta r_t(a'))}$$

$$p_{t+1}(a) = \frac{\exp\left(\boxed{\eta \sum_{i=1}^{t} r_i(a)}\right)}{\sum_{a' \in \mathcal{A}} \exp\left(\eta \sum_{i=1}^{t} r_i(a')\right)} \rightarrow \eta t \cdot \underline{\hat{r}_t(a)}$$

$$p_{t+1} = \operatorname*{argmax}_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

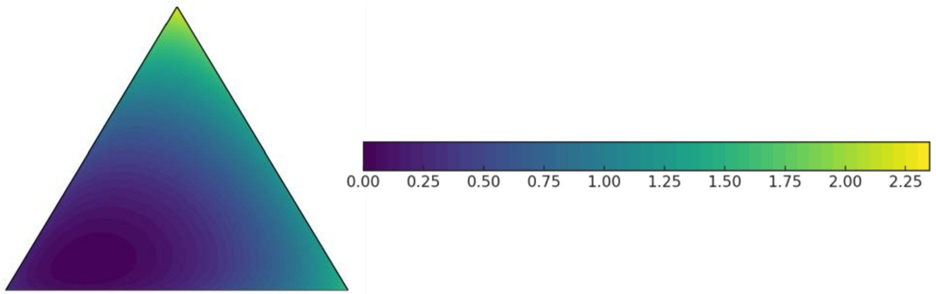
$$\mathrm{KL}(p, q) := \sum_{a=1}^{A} p(a) \ln \frac{p(a)}{q(a)}$$

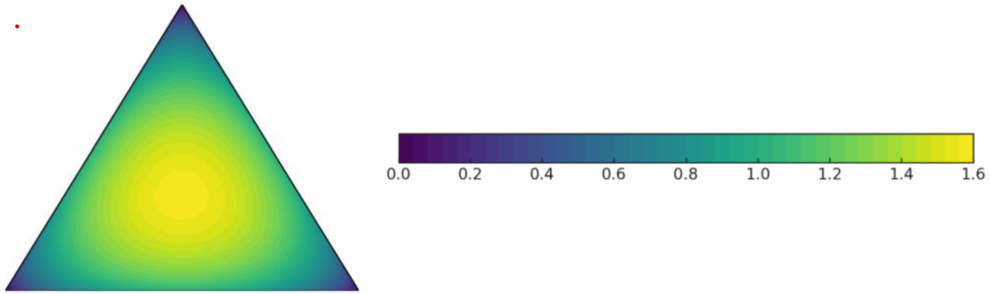
(KL divergence)

$$p_{t+1} = \operatorname*{argmax}_{p \in \Delta_{\mathcal{A}}} \left\{ \left\langle p, \sum_{i=1}^{t} r_i \right\rangle + \frac{1}{\eta} H(p) \right\}$$

$$H(p) := \sum_{a=1}^{A} p(a) \ln \frac{1}{p(a)}$$

(Shannon entropy)





Regret Bound for Exponential Weight Updates

Theorem.

Assume that $\eta r_t(a) \le 1$ for all t, a. Then EWU

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))}$$

ensures

$$\text{Regret} = \max_{a^\star} \sum_{t=1}^{T} \left(r_t(a^\star) - \langle p_t, r_t\rangle\right) \le \frac{\ln A}{\eta} + \eta \sum_{t=1}^{T}\sum_{a=1}^{A} p_t(a)\underline{r_t(a)^2}$$

$$|r_t(a)| \le R_{max}$$

$$\le \frac{\ln A}{\eta} + \eta \sum_t \underbrace{\left(\sum_a p_t(a)\right)}_{=1} R_{max}^2$$

$$= \frac{\ln A}{\eta} + \eta\, T \cdot R_{max}^2$$

$$\Big\downarrow \text{ optimal } \eta$$

$$R_{max}\sqrt{T \ln A}$$

Regret Bound Analysis

$$p_1(a) = \frac{1}{A}$$

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a'} p_t(a')\exp(\eta r_t(a'))}$$

$$\Rightarrow \log\frac{p_{t+1}(a^*)}{p_t(a^*)} = \log\frac{\exp(\eta r_t(a^*))}{\sum_a p_t(a)\exp(\eta r_t(a))} = \eta r_t(a^*) - \log\left(\sum_a p_t(a)\exp(\eta r_t(a))\right)$$

$$\Rightarrow \underline{r_t(a^*) - \langle p_t, r_t\rangle} = \frac{1}{\eta}\log\frac{p_{t+1}(a^*)}{p_t(a^*)} + \underbrace{\frac{1}{\eta}\log\left(\sum_a p_t(a)\exp(\eta r_t(a))\right) - \langle p_t, r_t\rangle}_{\boxed{\le}\ \eta \sum_a p_t(a)\, r_t(a)^2}$$

$$\Rightarrow \text{Regret} = \sum_{t=1}^{T}\left(r_t(a^*) - \langle p_t, r_t\rangle\right)$$

$$= \frac{1}{\eta}\log\frac{\overset{\le 1}{\boxed{p_{T+1}(a^*)}}}{\underset{\frac{1}{A}}{\boxed{p_1(a^*)}}} + \eta\sum_t\sum_a p_t(a)\, r_t(a)^2$$

$$\le \frac{1}{\eta}\log A + \eta\sum_{t,a} p_t(a)\, r_t(a)^2$$

$$\frac{1}{\eta}\log\left(\sum_a p_t(a)\,\underbrace{\exp\left(\eta\, r_t(a)\right)}\right) - \langle p_t, r_t\rangle$$

$$\le \frac{1}{\eta}\log\left(\sum_a p_t(a)\underbrace{\left(1+\eta\, r_t(a)+\eta^2 r_t(a)^2\right)}\right) - \langle p_t, r_t\rangle$$

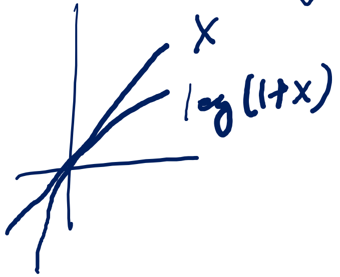
$e^x \le 1 + x + x^2$ for $x \le 1$

$$= \frac{1}{\eta}\log\left(\underbrace{\sum_a p_t(a)}_{1} + \eta\sum_a p_t(a)\, r_t(a) + \eta^2\sum_a p_t(a)\, r_t(a)^2\right) - \langle p_t, r_t\rangle$$

$$\le \frac{1}{\eta}\left(\eta\,\underline{\sum_a p_t(a)\, r_t(a)} + \eta^2\sum_a p_t(a)\, r_t(a)^2\right) \underline{- \langle p_t, r_t\rangle}$$

$\left[\ \underline{\log(1+x) \le x}\ \right]$ $\forall x > -1$

$$= \eta\sum_a p_t(a)\, r_t(a)^2$$



Exponential Weight Updates

$$\Delta_A = \{ X \in \mathbb{R}^A : \underline{\sum_a X(a) = 1}, \ \underline{X(a) \geq 0} \}$$

Exponential Weight Updates = KL divergence regularized policy updates

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))}$$

$=$

$$p_{t+1} = \underset{p \in \Delta_{\mathcal{A}}}{\operatorname{argmax}} \left\{ \underbrace{\langle p, r_t \rangle - \frac{1}{\eta}\mathrm{KL}(p, p_t)}_{F(p)} \right\}$$

$$\mathrm{KL}(P, P_t) = \sum_a p(a) \log \frac{p(a)}{P_t(a)}$$

$$\frac{\partial}{\partial p(a)} = \log \frac{p(a)}{P_t(a)} + 1$$

$$\mathcal{L}(p) = \underline{F(p)} + \underline{\lambda\Big(\sum_a p(a) - 1\Big)} + \underline{\sum_a u(a)\, p(a)}, \quad u(a) \geq 0$$

$$u(a)\,p(a) = 0 \quad \forall a$$

$$\frac{\partial}{\partial p(a)} = r_t(a) - \frac{1}{\eta}\Big(\log \frac{P(a)}{P_t(a)} + 1\Big) + \lambda + u(a) = 0$$

$$\Rightarrow \quad p(a) = P_t(a) \exp\Big(\eta r_t(a) + \eta\lambda + \text{[illegible]}\Big)$$

KL divergence regularized policy updates is the basis of many RL algorithms (e.g., PPO, SAC).

Projected Gradient Descent

Projected Gradient Descent = Euclidean norm regularized policy updates

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$= \quad p_{t+1} = \underset{p \in \Delta_{\mathcal{A}}}{\operatorname{argmax}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \|p - p_t\|_2^2 \right\}$$

Distance Regularized Updates

Projected Gradient Descent

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \|p - p_t\|_2^2 \right\}$$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

- Adversarial reward
- Stochastic reward
- For non-linear functions, gradient only approximates the function locally

General Framework: Mirror Descent

Projected Gradient Descent

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \|p - p_t\|_2^2 \right\}$$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

$$\psi(p) = \frac{1}{2} \|p\|_2^2$$

$$\psi(p) = \sum_{a=1}^{A} p(a) \ln p(a)$$

Mirror Descent

$$p_{t+1} = \max_{p \in \Omega} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} D_\psi(p, p_t) \right\}$$

$$D_\psi(p, q) := \psi(p) - \psi(q) - \langle \nabla \psi(q), p - q \rangle$$

(Bregman divergence w.r.t. the potential function / regularizer ψ)

Online Linear Optimization and Projected Gradient Descent

Given: Convex feasible set $\Omega \subseteq \mathbb{R}^d$

For time $t = 1, 2, \ldots, T$:

Learner chooses a point $w_t \in \Omega$

Environment reveals a reward vector $r_t \in \mathbb{R}^d$

$$\mathrm{Regret} = \max_{w \in \Omega} \sum_{t=1}^{T} \langle w, r_t \rangle - \sum_{t=1}^{T} \langle w_t, r_t \rangle$$

Projected Gradient Descent

Arbitrary $w_1 \in \Omega$

$w_{t+1} = \Pi_\Omega(w_t + \eta r_t)$

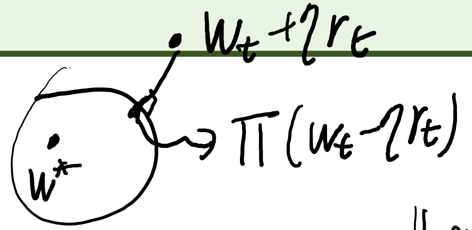
Regret Bound of Projected Gradient Descent

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$= \underline{DG\sqrt{T}}$$

Theorem. Projected Gradient Descent ensures

$$\sum_{t=1}^{T}\langle w^\star, r_t\rangle - \sum_{t=1}^{T}\langle w_t, r_t\rangle \le \frac{\|w^\star - w_1\|_2^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\|r_t\|_2^2 \le \overset{a}{\frac{D^2}{2\eta}} + \overset{b}{\frac{\eta G^2 T}{2}} \overset{\text{optimal}}{=} DG\sqrt{T}$$



$$\|w^* - w_{t+1}\|^2 = \|w^* - \Pi_\Omega(w_t + \eta r_t)\|^2$$

$$\le \|w^* - (w_t + \eta r_t)\|^2$$

$$= \|w^* - w_t\|^2 - 2\eta\langle w^* - w_t, r_t\rangle + \eta^2\|r_t\|^2$$

$$\|a+b\|^2$$

$$= \|a\|^2 + 2\langle a, b\rangle + \|b\|^2$$

$$\Rightarrow \sum_{t=1}^{T}\langle w^* - w_t, r_t\rangle \le \sum_{t=1}^{T}\frac{1}{2\eta}\left(\|w^* - w_t\|^2 - \|w^* - w_{t+1}\|^2\right) + \sum_{t=1}^{T}\frac{\eta}{2}\|r_t\|^2$$

$$\frac{1}{2\eta}\left(\|w^* - w_1\|^2 - \|w^* - w_{T+1}\|^2\right)$$

Summary

$\boxed{|r_t(a)| \le 1}$, $\forall a \Rightarrow \|r_t\| \doteq \sqrt{\sum_a (r_t(a))^2}$

$\approx \sqrt{A}$

Projected Gradient Descent

$$w_{t+1} = \Pi_{\Omega}(w_t + \eta r_t)$$

$$w_{t+1} = \max_{w \in \Omega} \left\{ \langle w, r_t \rangle - \frac{1}{2\eta} \|w - w_t\|_2^2 \right\}$$

$$\text{Regret} \le O\left(DG\sqrt{T}\right)$$

$$= O\left(\sqrt{AT}\right)$$ (in the expert setting)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$D = \max_{x,y \in \Omega} \|x - y\|_2$, $G = \max_t \|r_t\|_2$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

$$\text{Regret} \le O\left(\sqrt{T \log A}\right)$$